2 Summation of Arithmetic and Geometric Sequences

Activity

Activity 2.1 (p. 2.5)

(a) $S(5) = 1 + 3 + (5) + (7) + (9)$

$S(5) = 9 + (7) + (5) + (3) + (1)$

(b)

$$\begin{array}{rl} & S(5) = 1 + 3 + (5) + (7) + (9) \\ +) & S(5) = 9 + (7) + (5) + (3) + (1) \\ \hline & 2 \times S(5) = 10 + (10) + (10) + (10) + (10) \end{array}$$

(c) There are total (5) 10s in the above expression, therefore

$$2 \times S(5) = (5) \times 10$$
$$S(5) = \frac{(5) \times 10}{2} = \underline{\underline{25}}$$

(d)

$$S(9) = 1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 = \frac{(9) \times [(1) + (17)]}{2} = \underline{\underline{81}}$$

Activity 2.2 (p. 2.15)

(a) $S(5) = 2 + 2 \times (4) + 2 \times (4)^2 + 2 \times (4)^3 + 2 \times (4)^{(4)}$

(b) $4S(5) = 2 \times 4 + 2 \times (4)^{(2)} + 2 \times (4)^{(3)} + 2 \times (4)^{(4)}$ ...

(c)

$$\begin{array}{rl} & S(5) = 2 + 2 \times (4) + 2 \times (4)^2 + 2 \times (4)^3 + 2 \ldots \\ -) & 4S(5) = \quad 2 \times 4 + 2 \times (4)^{(2)} + 2 \times (4)^{(3)} + 2 \ldots \\ \hline & (1-4)S(5) = 2 + (0) + (0) + (0) + \ldots \end{array}$$

$$S(5) = \frac{(2) - 2 \times (4)^{(5)}}{1 - 4} = \frac{2[(1) - (4)^{(5)}]}{1 - 4} = \underline{\underline{682}}$$

(d) $$S(10) = \frac{(2)[(1) - (4)^{(10)}]}{(1) - (4)} = \underline{\underline{699\,050}}$$

Activity 2.3 (p. 2.23)

1.

r	n	r^n	Does the value of r^n get closer to zero as n increase? (✓/✗)
3	3	27	✗
	10	5.90×10^{4}	
	55	1.74×10^{26}	
	150	3.70×10^{71}	
$\frac{1}{2}$	3	0.125	✓
	10	9.77×10^{-4}	
	55	2.78×10^{-17}	
	150	7.01×10^{-46}	
−0.3	3	−0.027	✓
	10	5.90×10^{-6}	
	55	-1.74×10^{-29}	
	150	3.70×10^{-79}	
−2	3	−8	✗
	10	1024	
	55	-3.60×10^{16}	
	150	1.43×10^{45}	

2. ☐ $r < -1$ ☑ $-1 < r < 0$ ☑ $0 < r < 1$ ☐ $r > 1$

To Learn More

To Learn More (p. 2.28)

1. $$0.3\dot{1}\dot{5} = 0.315\,15\ldots = 0.3 + 0.015\,15\ldots = 0.3 + 0.015 + 0.000\,15 + 0.000\,001\,5 + \ldots$$

$0.0\dot{1}\dot{5}$ is a geometric series where the first term a = 0.015 and common ratio $r = \frac{0.000\,15}{0.015} = 0.01$.

Therefore,

$$0.3\dot{1}\dot{5} = 0.3 + \frac{0.015}{1 - 0.01} = 0.3 + \frac{0.015}{0.99} = \frac{3}{10} + \frac{1}{66} = \underline{\underline{\frac{52}{165}}}$$

2. $$0.\dot{3}2\dot{7} = 0.327\,327\ldots = 0.327 + 0.000\,327 + 0.000\,000\,327 + \ldots$$

It is a geometric series where the first term a = 0.327 and common ratio $r = \frac{0.000\,327}{0.327} = 0.001$.

Therefore,

$$0.\dot{3}2\dot{7} = \frac{0.327}{1 - 0.001} = \frac{0.327}{0.999} = \underline{\underline{\frac{109}{333}}}$$

Classwork

Classwork (p. 2.3)

1. **(a)** $$S(2) = T(1) + T(2) = 1 + 2 = \underline{\underline{3}}$$

(b) $$S(5) = T(1) + T(2) + T(3) + T(4) + T(5) = 1 + 2 + 4 + 8 + 16 = \underline{\underline{31}}$$

(c)

$$S(8) = T(1) + T(2) + T(3) + T(4) + T(5) + T(6) + T(7) + T(8)$$
$$= 1 + 2 + 4 + 8 + 16 + 32 + 64 + 128$$
$$= \underline{\underline{255}}$$

2. (a)
$$S(1) = T(1)$$
$$= \underline{\underline{10}}$$

(b)
$$S(4) = T(1) + T(2) + T(3) + T(4)$$
$$= 10 + 7 + 4 + 1$$
$$= \underline{\underline{22}}$$

(c)

$$S(7) = T(1) + T(2) + T(3) + T(4) + T(5) + T(6) + T$$
$$= 10 + 7 + 4 + 1 + (-2) + (-5) + (-8)$$
$$= \underline{\underline{7}}$$

Classwork (p. 2.25)

	Geometric sequence	Common ratio r	$S(\infty)$ exists or not?	Value of $S(\infty)$ (if exists)
(a)	24, 6, $\frac{3}{2}$, $\frac{3}{8}$, …	$\underline{\frac{1}{4}}$	yes	32
(b)	3, 6, 12, 24, …	2	no	/
(c)	$\frac{1}{4}$, $-\frac{1}{2}$, 1, –2, ...	–2	no	/
(d)	36, –18, 9, $-\frac{9}{2}$, …	$-\frac{1}{2}$	yes	24
(e)	10, 2, 0.4, 0.08, …	0.2	yes	12.5

Quick Practice

Quick Practice 2.1 (p. 2.4)

(a)
$$S(4) = T(1) + T(2) + T(3) + T(4)$$
$$= 1^3 + 2^3 + 3^3 + 4^3$$
$$= \underline{\underline{100}}$$

(b)

$$S(7) = T(1) + T(2) + T(3) + T(4) + T(5) + T(6) + T$$
$$= 1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3$$
$$= 784$$
$$\therefore \quad S(7) - S(4) = 784 - 100$$
$$= \underline{\underline{684}}$$

Alternative Solution

$$S(7) - S(4) = [T(1) + T(2) + T(3) + T(4) + T(5) + T($$
$$- [T(1) + T(2) + T(3) + T(4)]$$
$$= T(5) + T(6) + T(7)$$
$$= 5^3 + 6^3 + 7^3$$
$$= \underline{\underline{684}}$$

(c)

$$S(n+1) - S(n) = [T(1) + T(2) + \ldots + T(n+1)]$$
$$- [T(1) + T(2) + \ldots + T(n)]$$
$$= T(n+1)$$
$$= \underline{\underline{(n+1)^3}}$$

Quick Practice 2.2 (p. 2.8)

(a) Let a and d be the first term and the common difference of the sequence respectively.

∵ $a = -1$ and $d = -8 - (-1) = -7$

∴ The sum of the first 19 terms of the sequence:

$$S(19) = \frac{19}{2}[2(-1) + (19-1)(-7)]$$
$$= \underline{\underline{-1216}}$$

(b) Let a, d and ℓ be the first term, the common difference and the last term of the sequence respectively.
Suppose the kth term of the sequence is 31.

∵ $a = -2$, $d = 1 - (-2) = 3$, $\ell = T(k) = 31$

∴
$$31 = -2 + (k-1)(3)$$
$$31 = -5 + 3k$$
$$k = 12$$

∴ There are 12 terms in the arithmetic sequence.

∴
$$S(12) = \frac{12}{2}(-2 + 31)$$
$$= \underline{\underline{174}}$$

Quick Practice 2.3 (p. 2.9)

(a) (i) Let a and d be the first term and the common difference of the sequence respectively.

∵ $a = 5$ and $d = 2 - 5 = -3$

∴
$$S(15) = \frac{15}{2}[2(5) + (15-1)(-3)]$$
$$= \underline{\underline{-240}}$$

(ii)
$$S(40) = \frac{40}{2}[2(5) + (40-1)(-3)]$$
$$= \underline{\underline{-2140}}$$

(b) The sum from the 16th term to the 40th term of the sequence
$$= T(16) + T(17) + \ldots + T(40)$$
$$= S(40) - S(15)$$
$$= -2140 - (-240)$$
$$= \underline{\underline{-1900}}$$

Quick Practice 2.4 (p. 2.10)

(a) Let a and d be the first term and the common difference of the sequence respectively.

∵ $T(12) = -14$

∴
$$a + (12-1)d = -14$$
$$a + 11d = -14 \quad \ldots\ldots(1)$$

∵ $S(4) = 96$

∴ $\frac{4}{2}[2a+(4-1)d]=96$

$4a+6d=96$

$2a+3d=48$ ......(2)

2 × (1) – (2): $19d=-76$

$d=-4$

By substituting $d = -4$ into (1), we have

$a+11(-4)=-14$

$a=30$

∴ The first term is 30 and the common difference is –4.

(b) ∵ $a=30,\ d=-4$ and $S(k)=120$

∴ $\frac{k}{2}[2(30)+(k-1)(-4)]=120$

$\frac{k}{2}(64-4k)=120$

$16k-k^2=60$

$k^2-16k+60=0$

$(k-6)(k-10)=0$

$k=\underline{\underline{6}}$ or $k=\underline{\underline{10}}$

Quick Practice 2.5 (p. 2.11)

(a) The multiples of 4 between 100 and 250 inclusive are:

100, 104, 108, … , 248

They form an arithmetic sequence with first term 100 and common difference 4.

Let m be the number of terms in 100, 104, 108, … , 248.

$248=100+(m-1)(4)$

$m=38$

∴ The sum of all the multiples of 4 between 100 and 250 inclusive

$=\frac{38}{2}(100+248)$

$=\underline{\underline{6612}}$

(b) The multiples of 5 between 100 and 250 inclusive are:

100, 105, 110, … , 250

They form an arithmetic sequence with first term 100 and common difference 5.

Let n be the number of terms in 100, 105, 110, … , 250.

$250=100+(n-1)(5)$

$n=31$

∴ The sum of all the multiples of 5 between 100 and 250 inclusive

$=\frac{31}{2}(100+250)$

$=\underline{\underline{5425}}$

(c) The multiples of 20 between 100 and 250 inclusive are:

100, 120, 140, … , 240

They form an arithmetic sequence with first term 100 and common difference 20.

Let k be the number of terms in 100, 120, 140, … , 240.

$240=100+(k-1)(20)$

$k=8$

∴ The sum of all the multiples of 20 between 100 and 250 inclusive

$=\frac{8}{2}(100+240)$

$=1360$

∴ The sum of all the multiples of 4 or 5 between 100 and 250 inclusive

$=6612+5425-1360$

$=\underline{\underline{10\,677}}$

Quick Practice 2.6 (p. 2.17)

(a) Let a and r be the first term and the common ratio of the sequence respectively.

∵ $a = 1$ and $r=\frac{-5}{1}=-5$

∴ The sum of the first 7 terms of the sequence:

$S(7)=\frac{1[1-(-5)^7]}{1-(-5)}$

$=\underline{\underline{13\,021}}$

(b) Let a and r be the first term and the common ratio of the sequence respectively.

Suppose the kth term of the sequence is 3072.

∵ $a = 3,\ r=\frac{12}{3}=4$ and $T(k) = 3072$

∴ $3(4)^{k-1}=3072$

$4^{k-1}=1024$

$4^{k-1}=4^5$

$k-1=5$

$k=6$

∴ There are 6 terms in the geometric sequence.

$$\therefore \quad S(6) = \frac{3(4^6 - 1)}{4 - 1}$$
$$= 4096 - 1$$
$$= \underline{\underline{4095}}$$

Quick Practice 2.7 (p. 2.18)

(a) (i) Let a be the first term of the sequence.

$$\because \quad S(4) = 120$$
$$\therefore \quad \frac{a\left[1 - \left(\frac{1}{2}\right)^4\right]}{1 - \frac{1}{2}} = 120$$
$$a = 120 \times \frac{8}{15}$$
$$= 64$$

$\therefore$ The first term of the sequence is 64.

(ii)
$$S(10) = \frac{64\left[1 - \left(\frac{1}{2}\right)^{10}\right]}{1 - \frac{1}{2}}$$
$$= \frac{1023}{8}$$

$\therefore$ The sum from the 5th term to the 10th term of the sequence

$$= S(10) - S(4)$$
$$= \frac{1023}{8} - 120$$
$$= \underline{\underline{\frac{63}{8}}}$$

(b)
$$\because \quad S(m) = 127$$
$$\frac{64\left[1 - \left(\frac{1}{2}\right)^m\right]}{1 - \frac{1}{2}} = 127$$
$$\therefore \quad 128\left[1 - \left(\frac{1}{2}\right)^m\right] = 127$$
$$1 - \left(\frac{1}{2}\right)^m = \frac{127}{128}$$
$$\left(\frac{1}{2}\right)^m = \left(\frac{1}{2}\right)^7$$
$$m = \underline{\underline{7}}$$

Quick Practice 2.8 (p. 2.19)

Let a and r be the first term and the common ratio of the sequence respectively.

$$\because \quad a = 4 \text{ and } r = \frac{12}{4} = 3$$
$$\therefore \quad S(k) > 4000$$

$$\frac{4(3^k - 1)}{3 - 1} > 4000$$
$$3^k - 1 > 2000$$
$$3^k > 2001$$
$$\log 3^k > \log 2001$$
$$k \log 3 > \log 2001$$
$$k > \frac{\log 2001}{\log 3}$$
$$k > 6.9190...$$

∴ The minimum value of k is 7.

Quick Practice 2.9 (p. 2.20)

Let a be the first term of the sequence.

∵ $T(m) = -972$

∴ $a(-3)^{m-1} = -972$

$a(-3)^m = 2916$ ......(1)

∵ $S(m) = -728$

∴ $\frac{a[(-3)^m - 1]}{-3 - 1} = -728$

$a[(-3)^m - 1] = 2912$ ......(2)

$$\frac{a[(-3)^m - 1]}{a(-3)^m} = \frac{2912}{2916}$$
$$\frac{(-3)^m - 1}{(-3)^m} = \frac{728}{729}$$

$\frac{(2)}{(1)}$: $1 - \frac{1}{(-3)^m} = 1 - \frac{1}{729}$

$$\frac{1}{(-3)^m} = \frac{1}{(-3)^6}$$
$$(-3)^m = (-3)^6$$
$$m = \underline{\underline{6}}$$

Quick Practice 2.10 (p. 2.26)

Let a and r be the first term and the common ratio of the sequence respectively.

(a) ∵ $a = \frac{9}{2}$ and $r = \frac{3}{\frac{9}{2}} = \frac{2}{3}$

∴ The sum to infinity of the sequence:

$$S(\infty) = \frac{\frac{9}{2}}{1 - \frac{2}{3}} = \underline{\underline{\frac{27}{2}}}$$

(b) ∵ $a = 10$ and $r = \frac{-5}{10} = -\frac{1}{2}$

∴ The sum to infinity of the sequence:

$$S(\infty) = \frac{10}{1 - \left(-\frac{1}{2}\right)} = \frac{10}{\frac{3}{2}} = \underline{\underline{\frac{20}{3}}}$$

Quick Practice 2.11 (p. 2.26)

(a) Let r be the common ratio of the sequence.

$$r = \frac{\frac{a}{12}}{\frac{a}{4}} = \frac{1}{3}$$

∵ $S(\infty) = \frac{\frac{a}{4}}{1 - \frac{1}{3}}$

$$\therefore \quad \frac{\frac{a}{4}}{\frac{2}{3}} = \frac{27}{2}$$

$$a = \frac{27}{2} \times \frac{8}{3}$$
$$= \underline{\underline{36}}$$

(b) Let R be the common ratio of the sequence.

$$R = \frac{2y}{-2} = -y$$

$$\because \quad S(\infty) = \frac{-2}{1-(-y)}$$

$$\therefore \quad 3y - 4 = \frac{-2}{1+y}$$
$$(3y-4)(1+y) = -2$$
$$3y^2 - y - 2 = 0$$
$$(3y+2)(y-1) = 0$$
$$y = \underline{\underline{-\frac{2}{3}}} \text{ or } y = 1 \text{ (rejected)}$$

Quick Practice 2.12 (p. 2.27)

(a) $\because$ $2x + 3$, $2x$, $x + 1$, … are in geometric sequence.

$$\therefore \quad \frac{2x}{2x+3} = \frac{x+1}{2x}$$
$$4x^2 = 2x^2 + 5x + 3$$
$$2x^2 - 5x - 3 = 0$$
$$(2x+1)(x-3) = 0$$
$$x = \underline{\underline{-\frac{1}{2}}} \text{ or } x = \underline{\underline{3}}$$

(b) When $x = -\frac{1}{2}$, we find that the original geometric sequence is $2, -1, \frac{1}{2}, \dots$, where the first term is 2 and the common ratio is $\frac{-1}{2} = -\frac{1}{2}$.

The odd-numbered terms of the sequence, $T(1)$, $T(3)$, $T(5)$, … form another geometric sequence with first term 2 and common ratio $\left(-\frac{1}{2}\right)^2$.

$$\therefore \quad \text{The required sum} = \frac{2}{1-\left(-\frac{1}{2}\right)^2}$$
$$= \underline{\underline{\frac{8}{3}}}$$

When $x = 3$, we find that the original geometric sequence is $9, 6, 4, \dots$, where the first term is 9 and the common ratio is $\frac{6}{9} = \frac{2}{3}$.

The odd-numbered terms of the sequence, $T(1)$, $T(3)$, $T(5)$, … form another geometric sequence with first term 9 and common ratio $\left(\frac{2}{3}\right)^2$.

$\therefore$ The required sum $= \dfrac{9}{1-\left(\frac{2}{3}\right)^2}$

$= \underline{\underline{\dfrac{81}{5}}}$

Quick Practice 2.13 (p. 2.32)

(a) The numbers of boxes in successive layers form an arithmetic sequence with first term 12 and common difference 4.

Let $T(n)$ be the number of boxes in the nth layer.

From the question, we have

$$T(n) = 12 + (n-1)(4)$$
$$= 4n + 8$$

$\because$ There are 88 boxes in the last layer.

$\therefore$ $88 = 4n + 8$

$n = 20$

i.e. There are 20 layers of boxes.

$\therefore$ The total number of boxes $= \dfrac{20}{2}(12+88)$

$= \underline{\underline{1000}}$

(b) $\because$ $S(m) = \dfrac{m}{2}[2(12) + (m-1)(4)]$

$252 = 10m + 2m^2$

$\therefore$ $m^2 + 5m - 126 = 0$

$(m-9)(m+14) = 0$

$m = \underline{\underline{9}}$ or $m = -14$ (rejected)

Quick Practice 2.14 (p. 2.33)

(a) $\because$ The total saving at the end of February 2013 was $9802.

$\$5000 + \$5000(1 - r\%) = \$9802$

$5000(2 - r\%) = 9802$

$\therefore$ $2 - r\% = 1.9604$

$2 - \dfrac{r}{100} = 1.9604$

$r = \underline{\underline{3.96}}$

(b) The total saving at the end of 2013

$= \$5000 + \$5000(1-3.96\%) + \ldots + \$5000(1-3.96\%$

$= \dfrac{\$5000(1-0.9604^{12})}{1-0.9604}$

$= \underline{\underline{\$48\,513}}$ (cor. to the nearest dollar)

Quick Practice 2.15 (p. 2.34)

(a) The number of barrels of oil extracted in December 2012

$= 124\,000(1-4\%)^{11}$

$= \underline{\underline{79\,100}}$ (cor. to the nearest hundred)

(b) The total number of barrels of oil extracted in 2012

$= \dfrac{124\,000[1-(1-4\%)^{12}]}{1-(1-4\%)}$

$\approx 1\,200\,599.75$

The total number of barrels of oil extracted in 2013

$= \dfrac{124000(1-4\%)^{11}(1+1\%)[(1+1\%)^{12}-1]}{(1+1\%)-1}$

≈ 1013751.70

$\therefore$ The total number of barrels of oil extracted between January 2012 and December 2013 inclusive

$\approx 1\,200\,599.75 + 1\,013\,751.70$

$= \underline{\underline{2\,214\,400}}$ (cor. to the nearest hundred)

Quick Practice 2.16 (p. 2.36)

(a) Area of $\triangle A_1B_1C_1$

$= \dfrac{1}{2}(12)(12)\sin 60^\circ \text{ cm}^2$

$= \underline{\underline{36\sqrt{3} \text{ cm}^2}}$

$\because$ A_2 and B_2 are the mid-points of B_1C_1 and A_1C_1 respectively.

$\therefore$ $A_2B_2 = \dfrac{A_1B_1}{2}$ (mid-pt. theorem)

$= 6 \text{ cm}$

$\therefore$ Area of $\triangle A_2B_2C_2$

$= \dfrac{1}{2}(6)(6)\sin 60^\circ \text{ cm}^2$

$= \underline{\underline{9\sqrt{3} \text{ cm}^2}}$

(b) Similarly,

$A_nB_n = \dfrac{1}{2}A_{n-1}B_{n-1}$ (mid-pt. theorem)

$\therefore$

Area of $\triangle A_nB_nC_n$

$= \dfrac{1}{2}(A_nB_n)(A_nB_n)\sin 60^\circ \text{ cm}^2$

$= \dfrac{1}{2}\left(\dfrac{A_{n-1}B_{n-1}}{2}\right)\left(\dfrac{A_{n-1}B_{n-1}}{2}\right)\sin 60^\circ \text{ cm}^2$

$= \dfrac{1}{4}$ area of $\triangle A_{n-1}B_{n-1}C_{n-1}$

$\therefore$ The areas of the triangles form a geometric sequence with first term $36\sqrt{3}$ cm² and common ratio $\dfrac{1}{4}$.

$\therefore$ The sum of the areas of all the triangles formed

$= \dfrac{36\sqrt{3}}{1-\frac{1}{4}} \text{ cm}^2$

$= \underline{\underline{48\sqrt{3} \text{ cm}^2}}$

Further Practice

Further Practice (p. 2.11)

1. Let a, d and $T(n)$ be the first term, the common difference and the general term of the sequence respectively.

∵ $a = 158$ and $d = 153 - 158 = -5$

∴ $$T(n) = 158 + (n - 1)(-5) = 163 - 5n$$

Let $T(k)$ be the last positive term.

∵ $T(k) > 0$

∴ $$163 - 5k > 0$$
$$5k < 163$$
$$k < 32.6$$

∴ There are 32 positive terms.

∴ Sum of all the positive terms

$$= \frac{32}{2}[2(158) + (32 - 1)(-5)]$$
$$= \underline{\underline{2576}}$$

2. **(a)** Let a and d be the first term and the common difference of the sequence respectively.

$$T(8) = a + 7d = 2 \quad \text{......(1)}$$
$$T(12) = a + 11d = -6 \quad \text{......(2)}$$

(2) – (1): $$4d = -8$$
$$d = -2$$

By substituting $d = -2$ into (1), we have

$$a + 7(-2) = 2$$
$$a = 16$$

∴ The first term is 16 and the common difference is –2.

(b) Note that $T(2)$, $T(4)$, $T(6)$, … is an arithmetic sequence with first term $T(2) = 16 + (-2) = 14$ and common difference $= T(4) - T(2) = 2(-2) = -4$.

∵ $T(20)$ is the 10th term of the sequence.

∴

$$T(2) + T(4) + ... + T(20) = \frac{10}{2}[2(14) + (10 - 1)(-4)$$
$$= \underline{\underline{-40}}$$

3. **(a)** Let a and d be the first term and the common difference of the sequence respectively.

∵ $S(4) = 336$

∴ $$\frac{4}{2}[2a + (4 - 1)d] = 336$$
$$2a + 3d = 168 \quad \text{......(1)}$$

∵ $S(12) = 816$

∴ $$\frac{12}{2}[2a + (12 - 1)d] = 816$$
$$2a + 11d = 136 \quad \text{......(2)}$$

(2) – (1): $$8d = -32$$
$$d = -4$$

By substituting $d = -4$ into (1), we have

$$2a + 3(-4) = 168$$
$$a = 90$$

∴ The first term is 90 and the common difference is -4.

∴ The sum of the first 30 terms

$$= S(30)$$
$$= \frac{30}{2}[2(90) + (30 - 1)(-4)]$$
$$= \underline{\underline{960}}$$

(b) The sum of the first 40 terms

$$= S(40)$$
$$= \frac{40}{2}[2(90) + (40 - 1)(-4)]$$
$$= 480$$

∴ Sum from the 31st term to the 40th term

$$= S(40) - S(30)$$
$$= 480 - 960$$
$$= \underline{\underline{-480}}$$

Further Practice (p. 2.20)

1. **(a)** $T(1)=\frac{9}{2}(4)^{1-1}=\frac{9}{2}$

$T(2)=\frac{9}{2}(4)^{2-1}=18$

Common ratio $=\frac{T(2)}{T(1)}=\frac{18}{\frac{9}{2}}=4$

$S(n)=\frac{\frac{9}{2}(4^n-1)}{4-1}$

$=\underline{\underline{\frac{3}{2}(4^n-1)}}$

(b) $S(3)=\frac{3}{2}(4^3-1)=\frac{189}{2}$

$S(9)=\frac{3}{2}(4^9-1)=\frac{786\ 429}{2}$

$\therefore$ $T(4)+T(5)+T(6)+...+T(9)$

$=S(9)-S(3)$

$=\frac{786\ 429}{2}-\frac{189}{2}$

$=\underline{\underline{393\ 120}}$

2. $\because$ $T(1)=-1=(-1)^1(2)^{1-1}$

$T(2)=2=(-1)^2(2)^{2-1}$

$T(3)=-4=(-1)^3(2)^{3-1}$

$\vdots$

$\therefore$ $T(n)=(-1)^n(2)^{n-1}$

$T(1)$, $T(3)$, $T(5)$, $T(7)$, $T(9)$ are the first 5 negative terms.
Note that $T(1)$, $T(3)$, $T(5)$, $T(7)$, $T(9)$ form a geometric sequence with first term –1 and common ratio

$=\frac{T(3)}{T(1)}=\frac{-4}{-1}=4$.

$\therefore$ The sum of the first 5 negative terms

$=T(1)+T(3)+...+T(9)$

$=\frac{(-1)(4^5-1)}{4-1}$

$=\underline{\underline{-341}}$

3. **(a)** Let a be the first term of the sequence.

$\because$ $S(9)=4088$

$\therefore$ $\frac{a(2^9-1)}{2-1}=4088$

$a=8$

$\therefore$ The first term of the sequence is 8.

(b) $S(k)>10000$

$\frac{8(2^k-1)}{2-1}>10000$

$2^k-1>1250$

$2^k>1251$

$\log 2^k>\log 1251$

$k\log 2>\log 1251$

$k>\frac{\log 1251}{\log 2}$

$>10.2888...$

$\therefore$ The minimum value of k is 11.

Further Practice (p. 2.28)

1. (a) Let a and r be the first term and the common ratio of the sequence respectively.

$\because$ $a = 256$ and $r = \frac{-128}{256} = -\frac{1}{2}$

$\therefore$ $$S(\infty) = \frac{256}{1 - \left(-\frac{1}{2}\right)} = \underline{\underline{\frac{512}{3}}}$$

(b) The positive terms of the sequence, $T(1)$, $T(3)$, $T(5)$, … form another geometric sequence with first term 256 and common ratio $\left(-\frac{1}{2}\right)^2 = \frac{1}{4}$.

$\therefore$ The required sum $$= \frac{256}{1 - \frac{1}{4}} = \underline{\underline{\frac{1024}{3}}}$$

(c) The required sum $$= \frac{512}{3} - \frac{1024}{3} = \underline{\underline{-\frac{512}{3}}}$$

2. (a) $\because$ $1, \frac{1}{2-k}, \frac{1}{4+k}, \ldots$ are in geometric sequence.

$\therefore$
$$\left(\frac{1}{2-k}\right)^2 = 1 \times \left(\frac{1}{4+k}\right)$$
$$\frac{1}{4-4k+k^2} = \frac{1}{4+k}$$
$$4-4k+k^2 = 4+k$$
$$k^2-5k = 0$$
$$k(k-5) = 0$$
$$k = \underline{\underline{0}} \text{ or } k = \underline{\underline{5}}$$

(b) When $k = 0$, we find that the original geometric sequence is $1, \frac{1}{2}, \frac{1}{4}, \ldots$ where the first term is 1 and the common ratio is $\frac{\frac{1}{2}}{1} = \frac{1}{2}$.

$\therefore$ The required sum $$= \frac{1}{1-\frac{1}{2}} = \underline{\underline{2}}$$

When $k = 5$, we find that the original geometric sequence is $1, -\frac{1}{3}, \frac{1}{9}, \ldots$ where the first term is 1 and the common ratio is $\frac{-\frac{1}{3}}{1} = -\frac{1}{3}$.

$\therefore$ The required sum $$= \frac{1}{1-\left(-\frac{1}{3}\right)} = \underline{\underline{\frac{3}{4}}}$$

Exercise

Exercise 2A (p. 2.12)
Level 1

1. **(a)** $T(1) = \frac{1}{1} = \underline{\underline{1}}$

$T(2) = \underline{\underline{\frac{1}{2}}}$

$T(3) = \underline{\underline{\frac{1}{3}}}$

$T(4) = \underline{\underline{\frac{1}{4}}}$

$T(5) = \underline{\underline{\frac{1}{5}}}$

$T(6) = \underline{\underline{\frac{1}{6}}}$

(b) **(i)**

$$\begin{aligned} &S(6)\\ &= T(1) + T(2) + T(3) + T(4) + T(5) + T(6)\\ &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}\\ &= \underline{\underline{\frac{49}{20}}} \end{aligned}$$

(ii)

$$\begin{aligned} &S(6) - S(3)\\ &= T(4) + T(5) + T(6)\\ &= \frac{1}{4} + \frac{1}{5} + \frac{1}{6}\\ &= \underline{\underline{\frac{37}{60}}} \end{aligned}$$

2. **(a)** Let a and d be the first term and the common difference of the sequence respectively.

∵ $a = 4$ and $d = 8 - 4 = 4$

∴ $$\begin{aligned} S(20) &= \frac{20}{2}[2(4) + (20 - 1)(4)]\\ &= \underline{\underline{840}} \end{aligned}$$

(b) Let a and d be the first term and the common difference of the sequence respectively.

∵ $a = -5$ and $d = -2 - (-5) = 3$

∴ $$\begin{aligned} S(25) &= \frac{25}{2}[2(-5) + (25 - 1)(3)]\\ &= \underline{\underline{775}} \end{aligned}$$

(c) Let a and d be the first term and the common difference of the sequence respectively.

∵ $a = 78$ and $d = 65 - 78 = -13$

∴ $$\begin{aligned} S(15) &= \frac{15}{2}[2(78) + (15 - 1)(-13)]\\ &= \underline{\underline{-195}} \end{aligned}$$

(d) ∵ First term $= a + b$

and common difference $= (3a - b) - (a + b) = 2a - 2b$

∴

$$\begin{aligned} S(10) &= \frac{10}{2}[2(a + b) + (10 - 1)(2a - 2b)]\\ &= 5(20a - 16b)\\ &= \underline{\underline{100a - 80b}} \end{aligned}$$

3. **(a)** ∵ First term = 13, last term = 49
and number of terms = 7

∴ $$S(7) = \frac{7}{2}(13+49)$$
$$= \underline{\underline{217}}$$

(b) ∵ First term = 40, last term = –23
and number of terms = 8

∴ $$S(8) = \frac{8}{2}[40+(-23)]$$
$$= \underline{\underline{68}}$$

4. **(a)** **(i)** Suppose the *n*th term of the sequence is 300.
∵ First term = 9,
common difference = 12 – 9 = 3,
last term = $T(n)$ = 300

∴ $$300 = 9+(n-1)(3)$$
$$294 = 3n$$
$$n = 98$$

∴ The number of terms in the sequence is 98.

(ii) $$S(98) = \frac{98}{2}(9+300)$$
$$= \underline{\underline{15\,141}}$$

(b) **(i)** Suppose the *n*th term of the sequence is 95.
∵ First term = –1,
common difference = 2 – (–1) = 3,
last term = $T(n)$ = 95

∴ $$95 = -1+(n-1)(3)$$
$$99 = 3n$$
$$n = 33$$

∴ The number of terms in the sequence is 33.

(ii) $$S(33) = \frac{33}{2}[(-1)+95]$$
$$= \underline{\underline{1551}}$$

(c) **(i)** Suppose the *n*th term of the sequence is 1.
∵ First term = 49,
common difference = 45 – 49 = –4,
last term = $T(n)$ = 1

∴ $$1 = 49+(n-1)(-4)$$
$$-52 = -4n$$
$$n = 13$$

∴ The number of terms in the sequence is 13.

(ii) $$S(13) = \frac{13}{2}(49+1)$$
$$= \underline{\underline{325}}$$

(d) **(i)** Suppose the *n* th term of the sequence is 1 – 24*a*.
∵ First term = 1 – 2*a*,
common difference = (1 – 4*a*) – (1 – 2*a*) = –2*a*,
last term = $T(n)$ = 1 – 24*a*

∴ $$1-24a = 1-2a+(n-1)(-2a)$$
$$-24a = -2an$$
$$n = 12$$

∴ The number of terms in the sequence is 12.

(ii) $$S(12) = \frac{12}{2}[(1-2a)+(1-24a)]$$
$$= \underline{\underline{12-156a}}$$

5. **(a)** Let d be the common difference of the sequence.

∵ $S(10) = 325$

$$\frac{10}{2}[2(7)+(10-1)d]=325$$

∴ $$9d+14=65$$

$$d=\frac{17}{3}$$

∴ The common difference of the sequence is $\frac{17}{3}$.

(b) The 10th term

$$=T(10)$$
$$=7+(10-1)\left(\frac{17}{3}\right)$$
$$=\underline{\underline{58}}$$

6. **(a)** Let a be the first term of the sequence.

$$S(8)=\frac{8}{2}[2a+(8-1)(2)]$$
$$40=8a+56$$
$$a=-2$$

∴ The first term of the sequence is –2.

(b) $$S(20)=\frac{20}{2}[2(-2)+(20-1)(2)]$$
$$=\underline{\underline{340}}$$

7. **(a)** First term $=T(1)$ $=6-5(1)$ $=\underline{\underline{1}}$

$$T(2)=6-5(2)$$
$$=-4$$

Common difference $=T(2)-T(1)$ $=-4-1$ $=\underline{\underline{-5}}$

(b) $$S(12)=\frac{12}{2}[2(1)+(12-1)(-5)]$$
$$=\underline{\underline{-318}}$$

8. ∵ Common difference = 3 and the 2nd term = –10

∴ First term $=-10-3$ $=-13$

∴ $$S(12)=\frac{12}{2}[2(-13)+(12-1)(3)]$$
$$=\underline{\underline{42}}$$

9. **(a)** Let a and d be the first term and the common difference of the sequence respectively.

$T(4)=a+3d=55$ ...... (1)

$T(8)=a+7d=35$ ...... (2)

(2) − (1) : $4d=-20$

$d=-5$

By substituting $d=-5$ into (1), we have

$$a+3(-5)=55$$
$$a=70$$

∴ The first term is 70 and the common difference is –5.

(b) $S(10) = \frac{10}{2}[2(70) + (10 - 1)(-5)]$
$= \underline{\underline{475}}$

10. Let a and d be the first term and the common difference of the sequence respectively.
$\because \quad T(10) = 57$
$\therefore \quad a + (10 - 1)d = 57$
$a + 9d = 57 \quad ......(1)$
$\because \quad S(10) = 345$
$\therefore \quad \frac{10}{2}[2a + (10 - 1)d] = 345$
$2a + 9d = 69 ......(2)$
(2) – (1): $a = 12$
By substituting a = 12 into (1), we have
$12 + 9d = 57$
$d = 5$
$\therefore$ The first term is 12 and the common difference is 5.

11. First term = 9, common difference = 12 – 9 = 3
$\because \quad S(n) = 2100$
$\therefore \quad \frac{n}{2}[2(9) + (n-1)(3)] = 2100$
$3n^2 + 15n = 4200$
$n^2 + 5n - 1400 = 0$
$(n - 35)(n + 40) = 0$
$n = \underline{\underline{35}}$ or $n = -40$ (rejected)

12. First term = 96, common difference = 88 – 96 = –8
$\because \quad S(k) = 600$
$\therefore \quad \frac{k}{2}[2(96) + (k-1)(-8)] = 600$
$-8k^2 + 200k = 1200$
$k^2 - 25k + 150 = 0$
$(k - 10)(k - 15) = 0$
$k = \underline{\underline{10}}$ or $k = \underline{\underline{15}}$

13. Let a and d be the first term and the common difference of the sequence respectively.
(a) $\because \quad S(4) = 0$
$\therefore \quad \frac{4}{2}[2a + (4 - 1)d] = 0$
$2a + 3d = 0$
$a = -\frac{3}{2}d$
Take $d = 2$, $a = -\frac{3}{2}(2) = -3$
$\therefore$ The arithmetic sequence is –3, –1, 1, 3.
(or any other reasonable answers)

(b) $\because \quad S(5) = 0$
$\therefore \quad \frac{5}{2}[2a + (5 - 1)d] = 0$
$2a + 4d = 0$
$a = -2d$
Take $d = 2$, $a = -2(2) = -4$
$\therefore$ The arithmetic sequence is –4, –2, 0, 2, 4.
(or any other reasonable answers)

14. (a) ∵ First term = 20
and common difference = 16 – 20 = –4

∴ $T(n) = 20 + (n-1)(-4)$
$= \underline{\underline{24 - 4n}}$

(b) (i) ∵ $T(k) = -72$
∴ $24 - 4k = -72$
$k = \underline{\underline{24}}$

(ii) $S(24) = \frac{24}{2}[2(20) + (24-1)(-4)]$
$= \underline{\underline{-624}}$

15. (a) First term = –2, common difference = 1 – (–2) = 3

(i) $S(16) = \frac{16}{2}[2(-2) + (16-1)(3)]$
$= \underline{\underline{328}}$

(ii) $S(32) = \frac{32}{2}[2(-2) + (32-1)(3)]$
$= \underline{\underline{1424}}$

(b) The sum from the 17th term to the 32nd term of the sequence
$= S(32) - S(16)$
$= 1424 - 328$
$= \underline{\underline{1096}}$

Level 2

16. (a) Let a and d be the first term and the common difference of the sequence respectively.

∵ $S(14) = 406$
∴ $\frac{14}{2}[2a + (14-1)d] = 406$
$2a + 13d = 58$ ……(1)

∵ $T(4) + T(5) = 34$
∴ $(a + 3d) + (a + 4d) = 34$
$2a + 7d = 34$ ……(2)

(1) – (2): $6d = 24$
$d = 4$

By substituting $d = 4$ into (2), we have
$2a + 7(4) = 34$
$a = 3$

∴ The first term is 3 and the common difference is 4.

(b) $S(20) = \frac{20}{2}[2(3) + (20-1)(4)]$
$= \underline{\underline{820}}$

17. (a) Let a and d be the first term and the common difference of the sequence respectively.

∵ $S(5) = -270$
∴ $\frac{5}{2}[2a + (5-1)d] = -270$
$a + 2d = -54$ ……(1)

∵ $S(15) = -360$
∴ $\frac{15}{2}[2a + (15-1)d] = -360$
$a + 7d = -24$ ……(2)

(2) – (1): $5d = 30$
$d = 6$

By substituting $d = 6$ into (1), we have
$a + 2(6) = -54$
$a = -66$

∴ The first term is –66 and the common difference is 6.

(b) ∵ $S(m) = 234$
$\frac{m}{2}[2(-66) + (m-1)(6)] = 234$
∴ $6m^2 - 138m = 468$
$m^2 - 23m - 78 = 0$
$(m-26)(m+3) = 0$
$m = \underline{\underline{26}}$ or $m = -3$ (rejected)

18. (a) Let a and d be the first term and the common difference of the sequence respectively.

∵ $a = 99$ and $d = 92 - 99 = -7$
∴ $T(n) = 99 + (n-1)(-7)$
$= 106 - 7n$

Let $T(k)$ be the smallest positive term.
∵ $T(k) > 0$
$106 - 7k > 0$
∴ $106 > 7k$
$k < \frac{106}{7}$
$k < 15.1428...$

∴ The number of positive term in the sequence is 15.

(b) The sum of all the positive terms
$= S(15)$
$= \frac{15}{2}[2(99) + (15-1)(-7)]$
$= \underline{\underline{750}}$

19. (a) Let a and d be the first term and the common difference of the sequence respectively.

∵ $a = -122$ and $d = -109 - (-122) = 13$
∴ $T(n) = -122 + (n-1)(13)$
$= 13n - 135$

Let $T(k)$ be the largest negative term.
∵ $T(k) < 0$
$13k - 135 < 0$
∴ $k < \frac{135}{13}$
$k < 10.3846...$

∴ The largest negative term $= T(10)$
$= 13(10) - 135$
$= \underline{\underline{-5}}$

(b) $T(1)$, $T(3)$, $T(5)$, … are all the even terms in the sequence.

Note that $T(1)$, $T(3)$, $T(5)$, … form an arithmetic sequence with first term –122 and common difference = 2(13) = 26.

∵ $T(10)$ is the largest odd negative term.

∴ $T(9)$ is the largest even negative term and there are 5 terms in the sequence.

∴ The sum of all even negative terms

$$=T(1)+T(3)+...+T(9)$$
$$=\frac{5}{2}[2(-122)+(5-1)(26)]$$
$$=\underline{\underline{-350}}$$

20. (a) First term = $T(1)=5(1)-44=-39$

Common difference

$$=T(2)-T(1)$$
$$=5(2)-44-[5(1)-44]$$
$$=5$$

$$S(10)=\frac{10}{2}[2(-39)+(10-1)(5)]$$
$$=\underline{\underline{-165}}$$

(b)
$$S(22)=\frac{22}{2}[2(-39)+(22-1)(5)]$$
$$=297$$

∴ The sum from the 11th term to the 22nd term of the sequence

$$=S(22)-S(10)$$
$$=297-(-165)$$
$$=462$$
$$>0$$

∴ David's claim is correct.

21. (a) The integers between 200 and 500 inclusive that are divisible by 5 are 200, 205, 210, ..., 500.
They form an arithmetic sequence with first term 200 and common difference 5.
Let k be the number of terms in 200, 205, 210, ..., 500.

$$500=200+(k-1)(5)$$
$$k=61$$

∴ The sum of all the integers between 200 and 500 inclusive that are divisible by 5

$$=\frac{61}{2}(200+500)$$
$$=\underline{\underline{21\,350}}$$

(b) The integers between 200 and 500 inclusive that are divisible by 7 are 203, 210, 217, ..., 497.
They form an arithmetic sequence with first term 203 and common difference 7.
Let m be the number of terms in 203, 210, 217, ..., 497.

$$497=203+(m-1)(7)$$
$$m=43$$

∴ The sum of all the integers between 200 and 500 inclusive that are divisible by 7

$$=\frac{43}{2}(203+497)$$
$$=\underline{\underline{15\,050}}$$

(c) The integers between 200 and 500 inclusive that are divisible by both 5 and 7 (i.e. 35) are 210, 245, 280, ..., 490.
They form an arithmetic sequence with first term 210 and common difference 35.
Let n be the number of terms in 210, 245, 280, ..., 490.

$$490 = 210 + (n - 1)(35)$$
$$n = 9$$

∴ The sum of all the integers between 200 and 500 inclusive that are divisible by both 5 and 7

$$= \frac{9}{2}(210 + 490)$$
$$= \underline{\underline{3150}}$$

(d) The sum of all the integers between 200 and 500 inclusive that are divisible by either 5 or 7

$$= 21\,350 + 15\,050 - 3150$$
$$= \underline{\underline{33\,250}}$$

22. (a) The multiples of 8 between 50 and 250 inclusive are:

56, 64, 72, ..., 248

They form an arithmetic sequence with first term 56 and common difference 8.

Let *m* be the number of terms in 56, 64, 72, ..., 248.

$$248 = 56 + (m - 1)(8)$$
$$m = 25$$

∴ The sum of all the multiples of 8 between 50 and 250 inclusive

$$= \frac{25}{2}(56 + 248)$$
$$= \underline{\underline{3800}}$$

(b) The sum of all integers between 50 and 250 inclusive

$$= \frac{201}{2}(50 + 250)$$
$$= 30\,150$$

∴ The sum of all the integers between 50 and 250 inclusive that are not the multiples of 8

$$= 30\,150 - 3800$$
$$= \underline{\underline{26\,350}}$$

23. (a) Let *a* and *d* be the first term and the common difference of the sequence respectively.

∵ $S(6) = 124$

∴ $\frac{6}{2}[2a + (6 - 1)d] = 124$

$6a + 15d = 124$ ……(1)

∵ $S(9) = 132$

∴ $\frac{9}{2}[2a + (9 - 1)d] = 132$

$6a + 24d = 88$ ……(2)

(2) − (1): $9d = -36$

$d = -4$

By substituting $d = -4$ into (1), we have

$$6a + 15(-4) = 124$$
$$6a = 184$$
$$a = \frac{92}{3}$$

∴ The first term is $\frac{92}{3}$ and the common difference is −4.

(b) ∵ $S(k) > 104$

$$\frac{k}{2}\left[2\left(\frac{92}{3}\right)+(k-1)(-4)\right] > 104$$

∴ $$\frac{k}{2}\left(\frac{196}{3}-4k\right) > 104$$

$$49k - 3k^2 > 156$$

$$3k^2 - 49k + 156 < 0$$

$$(3k-13)(k-12) < 0$$

$$\frac{13}{3} < k < 12$$

∴ The minimum value of k such that $S(k)$ is greater than 104 is 5.

24. $$S(27) = \frac{27}{2}(7+111)$$
$$= 1593$$

∴ The sum of the 25 numbers inserted
$$= S(27) - 7 - 111$$
$$= 1593 - 118$$
$$= \underline{\underline{1475}}$$

25. (a)
$$2^n = 2 \times 2^2 \times 2^3 \times \ldots \times 2^{64}$$
$$2^n = 2^{(1+2+3+\ldots+64)}$$
$$2^n = 2^{\frac{64}{2}(1+64)}$$
$$n = \underline{\underline{2080}}$$

(b)
$$2^2 \times 2^4 \times 2^6 \times \ldots \times 2^{128}$$
$$= 2^{2(1+2+3+\ldots+64)}$$
$$= 2^{2(2080)}$$
$$= \underline{\underline{2^{4160}}}$$

26. (a) ∵ The 1st row has 1 number and each succeeding row has 1 more number than the previous row.
∴ The nth row has n numbers.

(b) There are n numbers in the n row. The first row has 1 number and each succeeding row has 1 more number than the previous row.
There are $\frac{n(n+1)}{2}$ numbers in the first n rows.

(c) ∵ Last term of the $\frac{n(n+1)}{2}$ numbers is $\frac{n(n+1)}{2}$.
∴ The sum of the numbers in the first n rows

$$= \frac{\frac{n(n+1)}{2}}{2}\left[1+\frac{n(n+1)}{2}\right]$$
$$= \frac{n(n+1)}{4}\left[\frac{2+n(n+1)}{2}\right]$$
$$= \underline{\underline{\frac{n(n+1)(n^2+n+2)}{8}}}$$

(d) The first term in the nth row $= \frac{n(n+1)}{2} - n + 1$

The last term in the nth row $= \frac{n(n+1)}{2}$

There are n numbers in the nth row.

∴ The sum of the numbers in the nth row

$$= \frac{n}{2}\left[\frac{n(n+1)}{2} - n + 1 + \frac{n(n+1)}{2}\right]$$
$$= \frac{n}{2}[n(n+1) - n + 1]$$
$$= \underline{\underline{\frac{n}{2}(n^2+1)}}$$

<u>Alternative solution</u>

The sum of the numbers in the first $(n - 1)$ rows

$$= \frac{(n-1)[(n-1)+1)][(n-1)^2+(n-1)+2]}{8}$$
$$= \frac{n(n-1)(n^2-n+2)}{8}$$

∴ The sum of the numbers in the nth row

$$= \frac{n(n+1)(n^2+n+2)}{8} - \frac{n(n-1)(n^2-n+2)}{8}$$
$$= \frac{n}{8}[(n^3+2n^2+3n+2)-(n^3-2n^2+3n-2)]$$
$$= \frac{n}{8}(4n^2+4)$$
$$= \underline{\underline{\frac{n(n^2+1)}{2}}}$$

Exercise 2B (p. 2.21)

Level 1

1. Let a and r be the first term and the common ratio of the sequence respectively.

(a) ∵ $a = 1$ and $r = \frac{2}{1} = 2$

∴ $S(10) = \frac{1(2^{10}-1)}{2-1}$

$= \underline{\underline{1023}}$

(b) ∵ $a = 27$ and $r = \frac{9}{27} = \frac{1}{3}$

$$\begin{aligned}\therefore\quad S(7) &= \frac{27\left[1-\left(\frac{1}{3}\right)^7\right]}{1-\frac{1}{3}}\\ &= \frac{27\left(1-\frac{1}{2187}\right)}{\frac{2}{3}}\\ &= \frac{2186}{81}\times\frac{3}{2}\\ &= \underline{\underline{\frac{1093}{27}}}\end{aligned}$$

2. Let a and r be the first term and the common ratio of the sequence respectively.

(a) Suppose the kth term is 4374.

$\because\quad a = 2,\ r = \frac{6}{2} = 3$ and $T(k) = 4374$

$$\begin{aligned}\therefore\quad 2(3)^{k-1} &= 4374\\ 3^{k-1} &= 2187\\ 3^{k-1} &= 3^7\\ k-1 &= 7\\ k &= 8\end{aligned}$$

$\therefore$ The number of terms is 8.

$$\begin{aligned}\therefore\quad S(8) &= \frac{2(3^8-1)}{3-1}\\ &= \frac{2(6561-1)}{2}\\ &= \underline{\underline{6560}}\end{aligned}$$

(b) Suppose the mth term is –2048.

$\because\quad a = 2,\ r = \frac{-8}{2} = -4$ and $T(m) = -2048$

$$\begin{aligned}\therefore\quad 2(-4)^{m-1} &= -2048\\ (-4)^{m-1} &= -1024\\ (-4)^{m-1} &= (-4)^5\\ m-1 &= 5\\ m &= 6\end{aligned}$$

$\therefore$ The number of terms is 6.

$$\begin{aligned}\therefore\quad S(6) &= \frac{2[1-(-4)^6]}{1-(-4)}\\ &= \frac{2(1-4096)}{5}\\ &= \underline{\underline{-1638}}\end{aligned}$$

3. $\because$ Common ratio $= \frac{4}{-1} = -4$

$$\begin{aligned}\therefore\quad S(6) &= \frac{-1[1-(-4)^6]}{1-(-4)}\\ &= \frac{-1(1-4096)}{5}\\ &= \underline{\underline{819}}\end{aligned}$$

4. First term = 4 and common ratio = 2

$$\begin{aligned}\because\quad S(k) &= 2044\\ \frac{4(2^k-1)}{2-1} &= 2044\\ \therefore\quad 2^k-1 &= 511\\ 2^k &= 512\\ 2^k &= 2^9\\ k &= \underline{\underline{9}}\end{aligned}$$

5. First term = 1 and common ratio $= \frac{4}{1} = 4$

$\because \quad S(m) = 5461$

$\frac{1(4^m - 1)}{4 - 1} = 5461$

$\therefore \quad 4^m - 1 = 16\,383$

$4^m = 16\,384$

$4^m = 4^7$

$m = \underline{\underline{7}}$

6. **(a)** Let a and r be the first term and the common ratio of the sequence respectively.

$\because \quad a = 648$ and $T(4) = 24$

$648r^3 = 24$

$\therefore \quad r^3 = \frac{1}{27}$

$r = \frac{1}{3}$

$\therefore$ The common ratio is $\frac{1}{3}$.

(b) $S(8) = \dfrac{648\left[1 - \left(\frac{1}{3}\right)^8\right]}{1 - \frac{1}{3}}$

$= \underline{\underline{\frac{26\,240}{27}}}$

7. **(a)** Let a and r be the first term and the common ratio of the sequence respectively.

$T(5) = ar^4 = -\frac{3}{4}$ ……(1)

$T(6) = ar^5 = -\frac{3}{8}$ ……(2)

$\frac{(2)}{(1)}: \quad \frac{ar^5}{ar^4} = \dfrac{-\frac{3}{8}}{-\frac{3}{4}}$

$r = \frac{1}{2}$

By substituting $r = \frac{1}{2}$ into (1), we have

$a\left(\frac{1}{2}\right)^4 = -\frac{3}{4}$

$a = -12$

$\therefore$ The first term and the common ratio are –12 and $\frac{1}{2}$ respectively.

(b) $S(6) = \dfrac{-12\left[1 - \left(\frac{1}{2}\right)^6\right]}{1 - \frac{1}{2}}$

$= \underline{\underline{-\frac{189}{8}}}$

8. (a) Let a be the first term of the sequence.

$\because$ $S(7) = -43$

$\therefore$
$$\frac{a\left[1-\left(-\frac{1}{2}\right)^7\right]}{1-\left(-\frac{1}{2}\right)} = -43$$
$$a\left(\frac{43}{64}\right) = -43$$
$$a = -64$$

$\therefore$ The first term is –64.

(b) $\because$ $S(k) = -\frac{5461}{128}$

$\therefore$
$$\frac{-64\left[1-\left(-\frac{1}{2}\right)^k\right]}{1-\left(-\frac{1}{2}\right)} = -\frac{5461}{128}$$
$$1-\left(-\frac{1}{2}\right)^k = \frac{16383}{16384}$$
$$\left(-\frac{1}{2}\right)^k = \left(-\frac{1}{2}\right)^{14}$$
$$k = \underline{\underline{14}}$$

9. (a) Suppose the k th term is x.

First term $= -\frac{1}{9}$ and common ratio $= \frac{\frac{1}{3}}{-\frac{1}{9}} = -3$

$\because$ $S(k) = -\frac{547}{9}$

$\therefore$
$$\frac{-\frac{1}{9}[(-3)^k - 1]}{-3-1} = -\frac{547}{9}$$
$$(-3)^k - 1 = -2188$$
$$(-3)^k = -2187$$
$$(-3)^k = (-3)^7$$
$$k = 7$$

$\therefore$ There are 7 terms in the sequence.

(b)
$$x = T(7)$$
$$= -\frac{1}{9}(-3)^{7-1}$$
$$= \underline{\underline{-81}}$$

10. First term $= T(1) = 2^{6-1} = 32$

and common ratio $= \frac{T(2)}{T(1)} = \frac{2^{6-2}}{2^{6-1}} = \frac{1}{2}$

$\therefore$
$$S(9) = \frac{32\left[1-\left(\frac{1}{2}\right)^9\right]}{1-\frac{1}{2}}$$
$$= \underline{\underline{\frac{511}{8}}}$$

11. First term = 4 and common ratio $= \frac{6}{4} = \frac{3}{2}$

$\because$ $S(k) > 800$

$\therefore$
$$\frac{4\left[\left(\frac{3}{2}\right)^k - 1\right]}{\frac{3}{2}-1} > 800$$
$$\left(\frac{3}{2}\right)^k > 101$$
$$\log\left(\frac{3}{2}\right)^k > \log 101$$
$$k\log\left(\frac{3}{2}\right) > \log 101$$
$$k > \frac{\log 101}{\log\left(\frac{3}{2}\right)}$$
$$k > 11.3822...$$

$\therefore$ The minimum value of k is 12.

12. First term = 32 and common ratio $= \frac{24}{32} = \frac{3}{4}$

$\because$ $S(m) < 124$

$\therefore$
$$\frac{32\left[1-\left(\frac{3}{4}\right)^m\right]}{\frac{1}{4}} < 124$$
$$1-\left(\frac{3}{4}\right)^m < \frac{31}{32}$$
$$\left(\frac{3}{4}\right)^m > \frac{1}{32}$$
$$\log\left(\frac{3}{4}\right)^m > \log\frac{1}{32}$$
$$m\log\left(\frac{3}{4}\right) > \log\frac{1}{32}$$
$$m < \frac{\log\left(\frac{1}{32}\right)}{\log\left(\frac{3}{4}\right)}$$
$$m < 12.0471...$$

$\therefore$ The greatest value of m is 12.

13.
$$a + a^2 = 2$$
$$a^2 + a - 2 = 0$$
$$(a+2)(a-1) = 0$$
$$a = -2 \text{ or } a = 1 \text{ (rejected)}$$

∴ Common ratio $= \dfrac{a^2}{a}$
$= a$
$= -2$

∴ $S(6) = \dfrac{-2[(-2)^6 - 1]}{-2-1}$
$= \underline{\underline{42}}$

Level 2

14. (a) Let a and r be the first term and the common ratio of the sequence respectively.

First term $= \dfrac{2}{3}$ and common ratio $= \dfrac{2}{\frac{2}{3}} = 3$

$$S(n) = \frac{\frac{2}{3}(3^n - 1)}{3-1}$$

∴ $= \dfrac{\frac{2}{3}(3^n - 1)}{2}$

$= \underline{\underline{\dfrac{1}{3}(3^n - 1)}}$

(b) The sum from the 3rd term to the 8th term

$= S(8) - S(2)$

$= \dfrac{1}{3}(3^8 - 1) - \dfrac{1}{3}(3^2 - 1)$

$= \dfrac{3^8 - 3^2}{3}$

$= \underline{\underline{2184}}$

15. (a) Let a and r be the first term and the common ratio of the sequence respectively.

$T(2) = ar = 48$ …… (1)

$T(6) = ar^5 = 768$ …… (2)

$\dfrac{(2)}{(1)}$: $\dfrac{ar^5}{ar} = \dfrac{768}{48}$

$r^4 = 16$

$r = 2$ or $r = -2$

By substituting $r = 2$ into (1), we have

$a(2) = 48$

$a = 24$

By substituting $r = -2$ into (1), we have

$a(-2) = 48$

$a = -24$

∴ The first term is 24 and the common ratio is 2 or the first term is –24 and the common ratio is –2.

(b) When $a = 24$ and $r = 2$,

$S(8) = \dfrac{24(2^8 - 1)}{2-1}$

$= \underline{\underline{6120}}$

When $a = -24$ and $r = -2$,

$S(8) = \dfrac{-24[1-(-2)^8]}{1-(-2)}$

$= \dfrac{-24(1-256)}{3}$

$= \underline{\underline{2040}}$

16. Let a be the first term of the sequence.

$\because \quad T(n) = \dfrac{6561}{16}$

$\therefore \quad a\left(\dfrac{3}{2}\right)^{n-1} = \dfrac{6561}{16}$

$a\left(\dfrac{3}{2}\right)^{n} = \dfrac{19\ 683}{32}$ ……(1)

$\because \quad S(n) = \dfrac{19\ 171}{16}$

$\therefore \quad \dfrac{a\left[\left(\dfrac{3}{2}\right)^{n} - 1\right]}{\dfrac{3}{2} - 1} = \dfrac{19\ 171}{16}$

$a\left[\left(\dfrac{3}{2}\right)^{n} - 1\right] = \dfrac{19\ 171}{32}$ ……(2)

$\dfrac{(2)}{(1)}$: $\quad \dfrac{a\left[\left(\dfrac{3}{2}\right)^{n} - 1\right]}{a\left(\dfrac{3}{2}\right)^{n}} = \dfrac{\dfrac{19\ 171}{32}}{\dfrac{19\ 683}{32}}$

$\dfrac{\left(\dfrac{3}{2}\right)^{n} - 1}{\left(\dfrac{3}{2}\right)^{n}} = \dfrac{19\ 171}{19\ 683}$

$1 - \dfrac{1}{\left(\dfrac{3}{2}\right)^{n}} = 1 - \dfrac{512}{19\ 683}$

$\dfrac{1}{\left(\dfrac{3}{2}\right)^{n}} = \left(\dfrac{2}{3}\right)^{9}$

$\left(\dfrac{2}{3}\right)^{n} = \left(\dfrac{2}{3}\right)^{9}$

$n = \underline{\underline{9}}$

17. Let r be the common ratio of the sequence.

$\because \quad T(n) = 192$

$\therefore \quad 3r^{n-1} = 192$

$r^{n-1} = 64$

$r^{n} = 64r$ ……(1)

$\because \quad S(n) = 381$

$\therefore \quad \dfrac{3(r^{n} - 1)}{r - 1} = 381$

$\dfrac{r^{n} - 1}{r - 1} = 127$ ……(2)

By substituting (1) into (2), we have

$\dfrac{64r - 1}{r - 1} = 127$

$64r - 1 = 127r - 127$

$63r = 126$

$r = 2$

By substituting $r = 2$ into (1), we have

$2^{n} = 64(2)$

$2^{n} = 2^{7}$

$n = \underline{\underline{7}}$

18. Let a and r be the first term and the common ratio of the sequence respectively.

$\because \quad S(4) = 20$

$\therefore \quad \dfrac{a(r^{4} - 1)}{r - 1} = 20$ ……(1)

$\because \quad S(8) = 20 + 320$

$\therefore \quad \dfrac{a(r^8 - 1)}{r - 1} = 340 \quad ......(2)$

$\dfrac{(2)}{(1)}: \quad \dfrac{\frac{a(r^8-1)}{r-1}}{\frac{a(r^4-1)}{r-1}} = \dfrac{340}{20}$

$\dfrac{r^8-1}{r^4-1} = 17$

$r^8 - 1 = 17r^4 - 17$

$(r^4)^2 - 17r^4 + 16 = 0$

$(r^4 - 1)(r^4 - 16) = 0$

$r^4 = 16 \text{ or } r^4 = 1$

$r = \pm 2 \text{ or } \quad r = \pm 1 \text{ (rejected)}$

By substituting $r = 2$ into (1), we have

$\dfrac{a(2^4 - 1)}{2 - 1} = 20$

$15a = 20$

$a = \dfrac{4}{3}$

By substituting $r = -2$ into (1), we have

$\dfrac{a[(-2)^4 - 1]}{-2 - 1} = 20$

$15a = -60$

$a = -4$

$\therefore$ The possible values of the first term are –4 or $\dfrac{4}{3}$.

19. (a) $\because$ First term = 16 384

and common ratio $= \dfrac{-4096}{16\,384} = -\dfrac{1}{4}$

$\therefore \quad S(8) = \dfrac{16\,384\left[1 - \left(-\frac{1}{4}\right)^8\right]}{1 - \left(-\frac{1}{4}\right)}$

$= \dfrac{\frac{65\,535}{4}}{\frac{5}{4}}$

$= \underline{\underline{13\,107}}$

(b) (i) 16 384, 1024, 64, ... are the positive terms.
Note that it is a geometric sequence with first term 16 384 and common ratio $= \dfrac{1024}{16\,384} = \dfrac{1}{16}$.

$\therefore$ The sum of the first 4 positive terms

$= \dfrac{16\,384\left[1 - \left(\frac{1}{16}\right)^4\right]}{1 - \frac{1}{16}}$

$= \dfrac{16\,384\left(\frac{65\,535}{65\,536}\right)}{\frac{15}{16}}$

$= \underline{\underline{17\,476}}$

(ii) The sum of the first 4 negative terms

$= 13\,107 - 17\,476$

$= \underline{\underline{-4369}}$

20. (a)

$x^2 = 72 \times 128$

$x^2 = 9216$

$x = \underline{\underline{\pm 96}}$

(b) Let r be the common ratio of the sequence.

$T(6) = 72r^5 > 0$

$\therefore$ r is a positve number.

$\therefore \quad r = \dfrac{96}{72}$

$= \dfrac{4}{3}$

$\because \quad S(k) > 1000$

$\dfrac{72\left[\left(\frac{4}{3}\right)^k - 1\right]}{\frac{4}{3} - 1} > 1000$

$\left(\dfrac{4}{3}\right)^k - 1 > \dfrac{125}{27}$

$\therefore \quad \left(\dfrac{4}{3}\right)^k > \dfrac{152}{27}$

$\log\left(\dfrac{4}{3}\right)^k > \log\dfrac{152}{27}$

$k\log\dfrac{4}{3} > \log\dfrac{152}{27}$

$k > \dfrac{\log\frac{152}{27}}{\log\frac{4}{3}}$

$k > 6.0067...$

$\therefore$ The minimum value of k is 7.

21. (a) Let a and r be the first term and the common ratio of the sequence respectively.

$a = S(1)$

$\because \quad = \frac{27}{2}\left[1 - \left(\frac{1}{3}\right)^1\right]$

$= 9$

$a + ar = S(2)$

$\therefore \quad 9 + 9r = \frac{27}{2}\left[1 - \left(\frac{1}{3}\right)^2\right]$

$9r = 3$

$r = \frac{1}{3}$

$\therefore \quad T(n) = 9\left(\frac{1}{3}\right)^{n-1}$

$= \underline{\underline{3^{3-n}}}$

(b) $T(5)$, $T(7)$, $T(9)$, ..., $T(15)$ is a geometric sequence with first term $T(5) = 3^{3-5} = \frac{1}{9}$ and common ratio

$= \frac{T(7)}{T(5)} = \frac{3^{3-7}}{3^{3-5}} = \frac{1}{9}$.

$\therefore$

$$T(5) + T(7) + T(9) + \ldots + T(15) = \frac{\frac{1}{9}\left[1 - \left(\frac{1}{9}\right)^6\right]}{1 - \frac{1}{9}}$$

$$= \frac{\frac{1}{9}\left(\frac{531440}{531441}\right)}{\frac{8}{9}}$$

$$= \frac{66430}{531441}$$

$$< 1$$

$\therefore$ Kenny's claim is correct.

22. Let a be the first term of the sequence.

$\because \quad T(n) = \frac{1}{6}$

$\therefore \quad a\left(\frac{1}{3}\right)^{n-1} = \frac{1}{6}$

$a\left(\frac{1}{3}\right)^n = \frac{1}{18} \quad$ ......(1)

$\because \quad S(n) = \frac{121}{6}$

$\therefore \quad \frac{a\left[1 - \left(\frac{1}{3}\right)^n\right]}{1 - \frac{1}{3}} = \frac{121}{6}$

$a\left[1 - \left(\frac{1}{3}\right)^n\right] = \frac{121}{9} \quad$ ......(2)

$$\frac{a\left[1 - \left(\frac{1}{3}\right)^n\right]}{a\left(\frac{1}{3}\right)^n} = \frac{\left(\frac{121}{9}\right)}{\left(\frac{1}{18}\right)}$$

$\frac{(2)}{(1)}: \quad \frac{1 - \left(\frac{1}{3}\right)^n}{\left(\frac{1}{3}\right)^n} = 242$

$\left(\frac{1}{3}\right)^{-n} - 1 = 242$

$3^n = 3^5$

$n = 5$

By substituting $n = 5$ into (1), we have

$$a\left(\frac{1}{3}\right)^5 = \frac{1}{18}$$
$$a = \frac{27}{2}$$

∴ The sum from the nth term to the $2n$th term
= The sum from the 5th term to the 10th term

$$= S(10) - S(4)$$
$$= \frac{\frac{27}{2}\left[1 - \left(\frac{1}{3}\right)^{10}\right]}{1 - \frac{1}{3}} - \frac{\frac{27}{2}\left[1 - \left(\frac{1}{3}\right)^{4}\right]}{1 - \frac{1}{3}}$$
$$= \frac{\frac{27}{2}\left[-\left(\frac{1}{3}\right)^{10} + \left(\frac{1}{3}\right)^{4}\right]}{1 - \frac{1}{3}}$$
$$= \frac{\frac{27}{2}\left(\frac{728}{59\,049}\right)}{\frac{2}{3}}$$
$$= \underline{\underline{\frac{182}{729}}}$$

23. Let a and r be the first term and the common ratio of the sequence respectively.

∵ $S(3) = 26$

∴ $\frac{a(r^3 - 1)}{r - 1} = 26$ ......(1)

∵ $S(6) - S(3) = 702$

∴ $\frac{a(r^6 - 1)}{r - 1} - 26 = 702$

$$\frac{a(r^6 - 1)}{r - 1} = 728 \quad \text{......(2)}$$

$\frac{(2)}{(1)}$:
$$\frac{\frac{a(r^6 - 1)}{r - 1}}{\frac{a(r^3 - 1)}{r - 1}} = \frac{728}{26}$$
$$\frac{r^6 - 1}{r^3 - 1} = 28$$
$$r^6 - 1 = 28r^3 - 28$$
$$(r^3)^2 - 28r^3 + 27 = 0$$
$$(r^3 - 27)(r^3 - 1) = 0$$
$$r^3 = 27 \text{ or } r^3 = 1$$
$$r = 3 \text{ or } r = 1 \text{ (rejected)}$$

By substituting $r = 3$ into (1), we have

$$\frac{a(3^3 - 1)}{3 - 1} = 26$$
$$a = 2$$

∴ $S(12) = \frac{2(3^{12} - 1)}{3 - 1}$
$$= \underline{\underline{531\ 440}}$$

24. (a) Let r be the common ratio of the sequence.

$$\because \quad S(3)=292$$
$$\frac{4(r^3-1)}{r-1}=292$$
$$\therefore \quad \frac{r^3-1}{r-1}=73$$
$$(r-1)(r^2+r+1)=73(r-1)$$
$$(r-1)[(r^2+r+1)-73]=0$$

$$(r-1)(r-8)(r+9)=0$$
$$r=1 \text{ (rejected) or } r=8 \text{ or } r=-9$$

∴ The possible values of the common ratio are 8 or –9.

(b) ∵ The common ratio is negative.

∴ $r=-9$

$$\therefore \quad S(6)=\frac{4[1-(-9)^6]}{1-(-9)}=\frac{4(1-531\,441)}{10}=\underline{\underline{-212\,576}}$$

25. (a)

$$a+a^3+a^5+\dots+a^{2n-1}=\frac{a[(a^2)^n-1]}{a^2-1}=\underline{\underline{\frac{a(a^{2n}-1)}{a^2-1}}}$$

(b)

$$3-9+3^3-9^3+3^5-9^5+\dots+3^{15}-9^{15}$$
$$=(3+3^3+3^5+\dots+3^{15})-(9+9^3+9^5+\dots+9^{15})$$
$$=(3+3^3+3^5+\dots+3^{2(8)-1})-(9+9^3+9^5+\dots+9^{2(8)-1})$$
$$=\frac{3(3^{16}-1)}{3^2-1}-\frac{9(9^{16}-1)}{9^2-1}$$
$$=\frac{3(3^{16}-1)}{3^2-1}-\frac{3^2(3^{32}-1)}{3^4-1}$$
$$=\frac{3(3^{16}-1)}{3^2-1}-\frac{3^2(3^{16}-1)(3^{16}+1)}{(3^2-1)(3^2+1)}$$
$$=\frac{3(3^{16}-1)(3^2+1)-3^2(3^{16}-1)(3^{16}+1)}{(3^2-1)(3^2+1)}$$
$$=\frac{[3(3^{16}-1)][10-3(3^{16}+1)]}{(8)(10)}$$
$$=-\frac{(3^{17}-3)(3^{17}-7)}{80}$$

Exercise 2C (p. 2.29)
Level 1

1. Let r be the common ratio of the sequence.

(a) ∵ $r=\frac{-12}{8}=-\frac{3}{2}<-1$

∴ No, the sum to infinity does not exist.

(b) ∵ $r=\frac{1}{5}$

∴ $-1<r<1$

∴ Yes, the sum to infinity exists.

(c) ∵ $r=\frac{-36}{-108}=\frac{1}{3}$

∴ $-1<r<1$

∴ Yes, the sum to infinity exists.

2. Let a and r be the first term and the common ratio of the sequence respectively.

(a) ∵ $a=9$ and $r=\frac{6}{9}=\frac{2}{3}$

$$\therefore \quad S(\infty)=\frac{9}{1-\frac{2}{3}}=\underline{\underline{27}}$$

(b) $\because$ $a = -4$ and $r = \frac{2}{-4} = -\frac{1}{2}$

$$S(\infty) = \frac{-4}{1-\left(-\frac{1}{2}\right)}$$

$\therefore$
$$= \frac{-4}{\frac{3}{2}}$$
$$= -\frac{8}{3}$$

(c) $\because$ $a = 1$ and $r = \frac{0.4}{1} = \frac{2}{5}$

$\therefore$
$$S(\infty) = \frac{1}{1-\frac{2}{5}}$$
$$= \frac{5}{3}$$

(d) $\because$ $a = 5$ and $r = \frac{-\frac{5}{3}}{5} = -\frac{1}{3}$

$$S(\infty) = \frac{5}{1-\left(-\frac{1}{3}\right)}$$

$\therefore$
$$= \frac{5}{\frac{4}{3}}$$
$$= \frac{15}{4}$$

3. Let r be the common ratio of the sequence.

$\because$ $S(\infty) = 8$

$\therefore$
$$\frac{3}{1-r} = 8$$
$$3 = 8 - 8r$$
$$8r = 5$$
$$r = \frac{5}{8}$$

$\therefore$ The common ratio of the sequence is $\frac{5}{8}$.

4. Let a be the first term of the sequence.

$\because$ $S(\infty) = 90$

$\therefore$
$$\frac{a}{1-0.2} = 90$$
$$\frac{a}{0.8} = 90$$
$$a = 72$$

$\therefore$ The first term of the sequence is 72.

5. Let a and r be the first term and the common ratio of the sequence respectively.

$$T(2) = 6$$
$$ar = 6$$
$$r = \frac{6}{a} \qquad \text{……(1)}$$

$\because$ $S(\infty) = 24$

$\therefore$ $\frac{a}{1-r} = 24 \qquad$ ……(2)

By substituting (1) into (2), we have

$$\frac{a}{1-\frac{6}{a}} = 24$$
$$a^2 = 24(a-6)$$
$$a^2 - 24a + 144 = 0$$
$$(a-12)^2 = 0$$
$$a = 12$$

$\therefore$ The first term of the sequence is 12.

6. Let a and r be the first term and the common ratio of the sequence respectively.

$T(4) = ar^3 = 3 \qquad$ ……(1)

$\because$ $S(\infty) = \frac{3}{4}a$

$\therefore$
$$\frac{a}{1-r} = \frac{3}{4}a$$
$$\frac{1}{1-r} = \frac{3}{4}$$
$$4 = 3 - 3r$$
$$3r = -1$$
$$r = -\frac{1}{3}$$

By substituting $r = -\frac{1}{3}$ into (1), we have

$$a\left(-\frac{1}{3}\right)^3 = 3$$
$$a\left(-\frac{1}{27}\right) = 3$$
$$a = -81$$

$\therefore$ The first term is –81 and the common ratio is $-\frac{1}{3}$.

7. Let a and r be the first term and the common ratio of the sequence respectively.

$$a = T(1) = 6^{3-1} = 36$$
$$T(2) = 6^{3-2} = 6$$
$$r = \frac{T(2)}{T(1)} = \frac{6}{36} = \frac{1}{6}$$

$\therefore$
$$S(\infty) = \frac{36}{1-\frac{1}{6}}$$
$$= \frac{216}{5}$$

8. (a) ∵ $k+1, 1, \frac{k}{2}, \ldots$ are in geometric sequence.

$$1^2 = (k+1)\times\left(\frac{k}{2}\right)$$
$$\therefore \quad 2 = k^2 + k$$
$$k^2 + k - 2 = 0$$
$$(k+2)(k-1) = 0$$
$$k = \underline{\underline{-2}} \text{ or } k = \underline{\underline{1}}$$

(b) When $k = -2$, we find that the original geometric sequence is $-1, 1, -1, \ldots$ where the first term is –1 and the common ratio is $\frac{1}{-1} = -1$.

∴ The sum to infinity does not exist.

When $k = 1$, we find that the original geometric sequence is $2, 1, \frac{1}{2}, \ldots$ where the first term is 2 and the common ratio is $\frac{1}{2}$.

∴ The sum to infinity $= \dfrac{2}{1-\frac{1}{2}} = \underline{\underline{4}}$

9. (a) Let a and r be the first term and the common ratio of the sequence respectively.

$$a = x \text{ and } r = \frac{-\frac{x}{6}}{x} = -\frac{1}{6}$$
$$\because \quad S(\infty) = -120$$
$$\therefore \quad \frac{x}{1-\left(-\frac{1}{6}\right)} = -120$$
$$x = \underline{\underline{-140}}$$

(b) First term = 1 and common ratio = r

$$\because \quad S(\infty) = 2r+1$$
$$\frac{1}{1-r} = 2r+1$$
$$1 = (2r+1)(1-r)$$
$$\therefore \quad 1 = -2r^2 + r + 1$$
$$2r^2 - r = 0$$
$$r(2r-1) = 0$$
$$r = \underline{\underline{\frac{1}{2}}} \text{ or } r = 0 \text{ (rejected)}$$

10. Let a and r be the first term and the common ratio of the sequence respectively.

$$\because \quad S(\infty) = \frac{3}{5}a$$
$$\frac{a}{1-r} = \frac{3}{5}a$$
$$\therefore \quad \frac{1}{1-r} = \frac{3}{5}$$
$$5 = 3 - 3r$$
$$3r = -2$$
$$r = -\frac{2}{3}$$

∴ The common ratio of the sequence is $-\frac{2}{3}$.

Level 2

11. The negative terms of the sequence –36, 30, –25, $\frac{125}{6}$, … form another geometric sequence with first term –36 and common ratio $\left(\frac{30}{-36}\right)^2 = \frac{25}{36}$.

$$\therefore \quad S(\infty) = \frac{-36}{1-\frac{25}{36}}$$
$$= \underline{\underline{-\frac{1296}{11}}}$$

12. Let a and r be the first term and the common ratio of the sequence respectively.

$$T(2) = ar = 12 \quad \ldots\ldots(1)$$
$$\because \quad S(\infty) = 64$$
$$\therefore \quad \frac{a}{1-r} = 64 \quad \ldots\ldots(2)$$

$$\frac{(1)}{(2)}: \quad \frac{ar}{\frac{a}{1-r}} = \frac{12}{64}$$
$$r(1-r) = \frac{3}{16}$$
$$16r - 16r^2 = 3$$
$$16r^2 - 16r + 3 = 0$$
$$(4r-1)(4r-3) = 0$$
$$r = \frac{1}{4} \text{ or } r = \frac{3}{4}$$

By substituting $r = \frac{1}{4}$ into (1), we have

$$a\left(\frac{1}{4}\right) = 12$$
$$a = 48$$

By substituting $r = \frac{3}{4}$ into (1), we have

$$a\left(\frac{3}{4}\right) = 12$$
$$a = 16$$

∴ The first term is 48 and the common ratio is $\frac{1}{4}$

or the first term is 16 and the common ratio is $\frac{3}{4}$.

13. Let a and r be the first term and the common ratio of the sequence respectively.

$$a + ar = 2$$
$$a(1+r) = 2$$
$$a = \frac{2}{1+r} \quad \ldots\ldots(1)$$

$\because \quad S(\infty) = 3$

$\therefore \quad \frac{a}{1-r} = 3 \quad \ldots\ldots(2)$

By substituting (1) into (2), we have

$$\frac{\frac{2}{1+r}}{1-r} = 3$$
$$\frac{2}{(1-r^2)} = 3$$
$$3 - 3r^2 = 2$$
$$3r^2 = 1$$
$$r^2 = \frac{1}{3}$$
$$r = \pm\frac{1}{\sqrt{3}}$$

$\therefore$ The common ratio is $\pm\frac{1}{\sqrt{3}}$ $\left(\text{or } \pm\frac{\sqrt{3}}{3}\right)$.

14. Let a and r be the first term and the common ratio of the sequence respectively.

$\because \quad S(3) = 21$

$\therefore \quad \frac{a(1-r^3)}{1-r} = 21 \quad \ldots\ldots(1)$

$\because \quad S(\infty) = 24$

$\therefore \quad \frac{a}{1-r} = 24 \quad \ldots\ldots(2)$

$\frac{(1)}{(2)}$:
$$1 - r^3 = \frac{7}{8}$$
$$r^3 = \frac{1}{8}$$
$$r = \frac{1}{2}$$

By substituting $r = \frac{1}{2}$ into (2), we have

$$\frac{a}{1-\frac{1}{2}} = 24$$
$$a = 12$$

$\therefore \quad T(8) = 12\left(\frac{1}{2}\right)^{8-1}$
$$= \underline{\underline{\frac{3}{32}}}$$

15. **(a)** Let a and r be the first term and the common ratio of the sequence respectively.

$\because \quad T(2) = 90$

$\therefore \quad ar = 90 \quad \ldots\ldots(1)$

$\because \quad T(5) = \frac{5}{12}$

$\therefore \quad ar^4 = \frac{5}{12} \quad \ldots\ldots(2)$

$\frac{(2)}{(1)}$:
$$\frac{ar^4}{ar} = \frac{\frac{5}{12}}{90}$$
$$r^3 = \frac{1}{216}$$
$$r = \frac{1}{6}$$

By substituting $r = \frac{1}{6}$ into (1), we have

$$a\left(\frac{1}{6}\right) = 90$$
$$a = 540$$

$\therefore \quad T(n) = \underline{\underline{540\left(\frac{1}{6}\right)^{n-1}}}$

(b) $S(2) = 540 + 90 = 630$

$$S(\infty) = \frac{540}{1-\frac{1}{6}} = 648$$

$$\begin{aligned} &T(3)+T(4)+T(5)+\dots \\ \therefore\ &= S(\infty)-S(2) \\ &= 648-630 \\ &= \underline{\underline{18}} \end{aligned}$$

16. (a)

$$\begin{aligned} T(1) &= a = 2^{1-1}ar^{1-1} \\ T(2) &= 2ar = 2^{2-1}ar^{2-1} \\ T(3) &= 4ar^2 = 2^{3-1}ar^{3-1} \end{aligned}$$

$\therefore\ T(n) = 2^{n-1}ar^{n-1}$

$$\frac{T(n)}{T(n-1)} = \frac{2^{n-1}ar^{n-1}}{2^{n-2}ar^{n-2}}$$

$= 2r$, which is a constant.

$\therefore$ a, $2ar$, $4ar^2$, … is a geometric sequence.

(b) For the sum to infinity exists,

$-1 < 2r < 1$

$\therefore\ -\frac{1}{2} < r < \frac{1}{2}$

$\therefore$ The required range of values of r is $-\frac{1}{2} < r < \frac{1}{2}$.

17. (a) Let a and r be the first term and the common ratio of the sequence respectively.

$\because$ $a = 1$ and $r = \frac{-\frac{1}{3}}{1} = -\frac{1}{3}$

$$\begin{aligned} S(\infty) &= \frac{1}{1-\left(-\frac{1}{3}\right)} \\ \therefore\ &= \frac{1}{\frac{4}{3}} \\ &= \underline{\underline{\frac{3}{4}}} \end{aligned}$$

(b) (i) $\because$ Each term in the sequence is the negative of the corresponding term in the sequence in (a).

$\therefore\ S(\infty) = \underline{\underline{-\frac{3}{4}}}$

(ii) $\because$ Each term in the sequence is the product of $-\frac{1}{6}$ and the corresponding term in the sequence in (a).

$$\begin{aligned} \therefore\ S(\infty) &= \frac{3}{4}\times\left(-\frac{1}{6}\right) \\ &= \underline{\underline{-\frac{1}{8}}} \end{aligned}$$

18. (a) Let a and r be the first term and the common ratio of the sequence respectively.

$$\begin{aligned} \because\ r &= \frac{T(n)}{T(n-1)} \\ &= \frac{3(2k-1)^n}{3(2k-1)^{n-1}} \\ &= 2k-1 \end{aligned}$$

$$\begin{aligned} \therefore\ -1 &< 2k-1 < 1 \\ 0 &< 2k < 2 \\ 0 &< k < 1 \end{aligned}$$

$\therefore$ The required range of values of k is $0 < k < 1$.

(b)

$$\begin{aligned} \because\ a &= T(1) \\ &= 3(2k-1)^1 \\ &= 3(2k-1) \end{aligned}$$

$\therefore\ S(\infty) = \frac{3(2k-1)}{1-(2k-1)} = \underline{\underline{\frac{3(2k-1)}{2(1-k)}}}$

(c) Suppose that $S(\infty) = 1$.

$$\begin{aligned} \frac{3(2k-1)}{2(1-k)} &= 1 \\ 6k-3 &= 2-2k \\ 8k &= 5 \\ k &= \frac{5}{8} \end{aligned}$$

$\because\ 0 < \frac{5}{8} < 1$

$\therefore$ It is possible that the sum of infinity of the sequence equals to 1.

19. (a) $\because$ $a, b, 7$ form an arithmetic sequence.

$\therefore\ 2b = 7 + a$

$a = 2b - 7$ ……(1)

$\because$ $b, -2, a$ form a geometric sequence.

$\therefore\ (-2)^2 = ab$

$ab = 4$ ……(2)

By substituting (1) into (2), we have

$$\begin{aligned} (2b-7)b &= 4 \\ 2b^2-7b-4 &= 0 \\ (2b+1)(b-4) &= 0 \end{aligned}$$

$b = \underline{\underline{4}}$ or $b = -\frac{1}{2}$ (rejected)

By substituting $b = 4$ into (1), we have

$$\begin{aligned} a &= 2(4)-7 \\ &= \underline{\underline{1}} \end{aligned}$$

(b) (i) From (a), we find that the original geometric sequence is $4, -2, 1, \dots$ where the first term is 4 and the common ratio is $\frac{-2}{4} = -\frac{1}{2}$.

$$\therefore \quad \text{The sum to infinity} = \frac{4}{1-\left(-\frac{1}{2}\right)} = \underline{\underline{\frac{8}{3}}}$$

(ii) The negative terms of the sequence, $T(2)$, $T(4)$, $T(6)$, … form another geometric sequence with first term –2 and common ratio $\left(-\frac{1}{2}\right)^2 = \frac{1}{4}$.

$$\therefore \quad \text{The required sum} = \frac{-2}{1-\frac{1}{4}} = \underline{\underline{-\frac{8}{3}}}$$

20. (a) (i) Let a and r be the first term and the common ratio of the sequence respectively.

$$T(5) = ar^4 = 48 \quad \text{……(1)}$$
$$T(8) = ar^7 = 6 \quad \text{……(2)}$$

$$\frac{(2)}{(1)}: \quad r^3 = \frac{1}{8}$$
$$r = \frac{1}{2}$$

By substituting $r = \frac{1}{2}$ into (1), we have

$$a\left(\frac{1}{2}\right)^4 = 48$$
$$a = 768$$
$$\therefore \quad \underline{\underline{T(n) = 768\left(\frac{1}{2}\right)^{n-1}}}$$

(ii)

$$S(\infty) = \frac{768}{1-\frac{1}{2}} = \underline{\underline{1536}}$$

(b) (i) Let t_1, t_2, t_3, … be the sequence $T(1)$, $\frac{1}{2}T(2)$, $\left(\frac{1}{2}\right)^2 T(3)$, ...

$$t_1 = T(1) = \left(\frac{1}{2}\right)^{1-1} T(1)$$
$$t_2 = \frac{1}{2}T(2) = \left(\frac{1}{2}\right)^{2-1} T(2)$$
$$t_3 = \left(\frac{1}{2}\right)^2 T(3) = \left(\frac{1}{2}\right)^{3-1} T(3)$$
$$\therefore \quad t_n = \left(\frac{1}{2}\right)^{n-1} T(n)$$

$$\frac{t_n}{t_{n-1}} = \frac{\left(\frac{1}{2}\right)^{n-1} T(n)}{\left(\frac{1}{2}\right)^{n-2} T(n-1)} = \left(\frac{1}{2}\right)\frac{T(n)}{T(n-1)} = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}, \text{ which is a constant.}$$

$\therefore$ $T(1)$, $\frac{1}{2}T(2)$, $\left(\frac{1}{2}\right)^2 T(3)$, $\left(\frac{1}{2}\right)^3 T(4)$, … is a geometric sequence.

$$S(\infty)=\frac{T(1)}{1-\frac{1}{4}}$$

(ii)
$$=\frac{768}{\frac{3}{4}}$$
$$=\underline{\underline{1024}}$$

21. Let a and r be the first term and the common ratio of the sequence respectively.

∵ $S(\infty)=21$

∴ $\frac{a}{1-r}=21$ ......(1)

All the odd-numbered terms $T(1)$, $T(3)$, $T(5)$, … are in geometric sequence with common ratio r^2.

∵ Sum to infinity of all the odd-numbered terms $=\frac{63}{4}$

∴ $\frac{a}{1-r^2}=\frac{63}{4}$ ....(2)

$\frac{(1)}{(2)}$:
$$\frac{1-r^2}{1-r}=\frac{4}{3}$$
$$4(1-r)=3(1-r^2)$$
$$4-4r=3-3r^2$$
$$3r^2-4r+1=0$$
$$(3r-1)(r-1)=0$$
$$r=\frac{1}{3} \quad \text{or} \quad r=1 \text{ (rejected)}$$

By substituting $r=\frac{1}{3}$ into (1), we have

$$\frac{a}{1-\frac{1}{3}}=21$$
$$a=14$$

∴
$$T(6)=14\left(\frac{1}{3}\right)^{6-1}$$
$$=\underline{\underline{\frac{14}{243}}}$$

Exercise 2D (p. 2.36)

Level 1

1. ∵ The number of cans in each layer is 3 less than the previous layer.

∴ The numbers of cans in successive layers form an arithmetic sequence with first term 40 and common difference −3.

∴ Total number of cans

$$=\frac{10[2(40)+(10-1)(-3)]}{2}$$
$$=\underline{\underline{265}}$$

2. ∵ The amount of money Angel saves in each week is \$10 less than the preceding week.

∴ The amount of money Angel saves in successive weeks form an arithmetic sequence with first term \$100 and common difference −\$10.

∴ Total amount of money Angel saves after 7 weeks

$$=\$\frac{7}{2}[2(100)+(7-1)(-10)]$$
$$=\$490$$
$$>\$480$$

∴ Angel will have enough money to buy the watch after 7 weeks.

3. Let $T(n)$ be the number of students in the nth row with first term 3 and common difference d.

(a) Suppose there are k rows of students.

$$S(k) = 210$$
$$\frac{k}{2}(3+39) = 210$$
$$k = 10$$

∴ There are 10 rows of students.

(b)
$$T(10) = 39$$
$$3+(10-1)d = 39$$
$$d = 4$$

∴ $T(n) = 3+(n-1)(4) = 4n-1$

∴ The number of students in the 6th row

$$= T(6) = 4(6)-1 = \underline{\underline{23}}$$

4. The amount received at the end of each year forms a geometric sequence with first term \$5000(1 + 6%) and common ratio (1 + 6%).

∴ The total amount received at the end of the 10th year

$$= \$\frac{5000(1+6\%)[(1+6\%)^{10}-1]}{(1+6\%)-1}$$
$$= \$\frac{5000(1.06)(1.06^{10}-1)}{0.06}$$
$$= \underline{\underline{\$69\,858}} \text{ (cor. to the nearest dollar)}$$

5. The amount of money in James' bank account at the end of each year forms a geometric sequence with first term \$4000(1 + 3.5%) and common ratio (1 + 3.5%).

∴ The required increase in the amount of money

$$= S(10) - S(5)$$
$$= \$\left[\frac{4000(1+3.5\%)[(1+3.5\%)^{10}-1]}{(1+3.5\%)-1} - \frac{4000(1+3.5\%)[(1+3.5\%)^{5}-1]}{(1+3.5\%)-1}\right]$$
$$= \$\frac{4000(1.035)(1.035^{10}-1.035^5)}{0.035}$$
$$= \underline{\underline{\$26\,367}} \text{ (cor. to the nearest dollar)}$$

6. (a) Let b cm be the length of the shortest portion. The lengths of the portions form an arithmetic sequence.

$$\frac{20}{2}(78+b) = 800$$
$$b = 2$$

∴ The length of the shortest portion is 2 cm.

(b)
$$2 = 78+(20-1)(-l)$$
$$l = \underline{\underline{4}}$$

7. (a) The maximum height the ball can reach forms a geometric sequence with first term $6\left(\frac{5}{7}\right)$ m and common ratio $\frac{5}{7}$.

∴ The maximum height the ball can reach in the nth rebound

$$= 6\left(\frac{5}{7}\right)\left(\frac{5}{7}\right)^{n-1} \text{ m}$$
$$= \underline{\underline{6\left(\frac{5}{7}\right)^{n} \text{ m}}}$$

(b) Suppose the ball reaches a height less than 1.5 m in the kth rebound.

$$T(k) < 1.5$$
$$6\left(\frac{5}{7}\right)^k < 1.5$$
$$\left(\frac{5}{7}\right)^k < \frac{1}{4}$$
$$\log\left(\frac{5}{7}\right)^k < \log\frac{1}{4}$$
$$k\log\left(\frac{5}{7}\right) < \log\frac{1}{4}$$
$$k > \frac{\log\frac{1}{4}}{\log\frac{5}{7}}$$
$$k > 4.1200...$$

Since k is an integer, $k = 5$.

∴ The ball will reach a height less than 1.5 m in at least 5 rebounds.

(c) The distances travelled in successive upwards (or downwards) are in geometric sequence with common ratio $\frac{5}{7}$.

∴ The total distance travelled by the ball before it stops

$$= \left[6 + 2\times\frac{6\left(\frac{5}{7}\right)}{1-\frac{5}{7}}\right] \text{m}$$
$$= \underline{\underline{36 \text{ m}}}$$

8. (a) (i) Perimeter of $T_1 = 4$ cm

Perimeter of $T_2 = 10 \text{ cm} = (4+6)$ cm

Perimeter of $T_3 = 16 \text{ cm} = [4+2(6)]$ cm

⋮

Perimeter of $T_n = [4+6(n-1)] \text{ cm} = (6n-2)$ cm

Perimeter of T_n – perimeter of T_{n-1}

$= (6n - 2)\text{ cm} - [6(n-1) - 2]\text{ cm}$

$= 6\text{ cm, which is a constant.}$

∴ The perimeters of T_1, T_2, T_3, … form an arithmetic sequence.

(ii) The sum of the perimeters of the first 15 figures

$$= \frac{15[2(4) + (15-1)(6)]}{2}\text{ cm}$$

$$= \underline{\underline{690\text{ cm}}}$$

(b) Area of $T_1 = 1\text{ cm}^2 = 1^2\text{ cm}^2$

Area of $T_2 = 4\text{ cm}^2 = 2^2\text{ cm}^2$

Area of $T_3 = 9\text{ cm}^2 = 3^2\text{ cm}^2$

⋮

Area of $T_n = n^2\text{ cm}^2$

∴ The sum of the areas of the first 9 figures

$$= (1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2)\text{ cm}^2$$

$$= \underline{\underline{285\text{ cm}^2}}$$

9. **(a)** The perimeters of the squares form a geometric sequence.

$T_1 = 4 \times 10\text{ cm} = 40\text{ cm}$

∵

$$A_2B_2^{\ 2} = B_1B_2^{\ 2} + B_1A_2^{\ 2} \quad \text{(Pyth. theorem)}$$

$$A_2B_2 = \sqrt{2\left(\frac{10}{2}\right)^2}\text{ cm}$$

$$= 5\sqrt{2}\text{ cm}$$

∴ $T_2 = 4 \times 5\sqrt{2}\text{ cm} = 20\sqrt{2}\text{ cm}$

∴ Common ratio $= \dfrac{T_2}{T_1} = \dfrac{20\sqrt{2}\text{ cm}}{40\text{ cm}} = \dfrac{\sqrt{2}}{2}$

∴ $T_n = \underline{\underline{40\left(\dfrac{\sqrt{2}}{2}\right)^{n-1}\text{ cm}}}$

(b) T_1, T_3, T_5, … form another geometric sequence with first term 40 cm and common ratio $= \left(\dfrac{\sqrt{2}}{2}\right)^2 = \dfrac{1}{2}$.

$$T_1 + T_3 + T_5 + \ldots$$

∴ $$= \frac{40}{1 - \frac{1}{2}}\text{ cm}$$

$$= \underline{\underline{80\text{ cm}}}$$

Level 2

10. **(a)** Let T_n be the number of blocks in the nth row.

∵ T_1, T_2, T_3, ... is an arithmetic sequence with first term 2 and common difference 2.

∴ The total number of blocks used

$$= S(20)$$

$$= \frac{20[2(2) + (20-1)(2)]}{2}$$

$$= \underline{\underline{420}}$$

(b) T_1, T_4, T_7, … , T_{19} is another arithmetic sequence with first term 2 and common difference = 3(2) = 6.
There are 7 terms in the sequence.

∴ Number of red blocks

$$= T_1 + T_4 + T_7 + \ldots + T_{19}$$

$$= \frac{7[2(2) + (7-1)(6)]}{2}$$

$$= \underline{\underline{140}}$$

∴ Number of white blocks

$= S(20) - (T_1 + T_4 + T_7 + ... + T_{19})$

$= 420 - 140$

$= \underline{\underline{280}}$

11. **(a)** **(i)** The numbers of seats in successive rows form an arithmetic sequence with first term 12 and common difference 3.

Let $T(k)$ be the number of seats in the last row.

$$\because \quad T(k) = 12 \times 6$$

$$12 + (k-1)(3) = 72$$

$$\therefore \quad 9 + 3k = 72$$

$$k = 21$$

∴ There are total 21 rows of seats.

(ii) Total number of seats $= \frac{21}{2}(12 + 72)$

$= \underline{\underline{882}}$

(b) Suppose the seat numbered 369 is located in the mth row.

$$S(m) \ge 369$$

$$\frac{m}{2}[2(12) + (m-1)(3)] \ge 369$$

$$3m^2 + 21m \ge 738$$

$$m^2 + 7m - 246 \ge 0$$

$$\therefore \quad m \ge \frac{-7 + \sqrt{7^2 - 4(1)(-246)}}{2(1)}$$

or

$$m \le \frac{-7 - \sqrt{7^2 - 4(1)(-246)}}{2(1)} \quad \text{(rejected)}$$

$$\therefore \quad m \ge 12.5701...$$

Since m is an integer, the minimum value of m is 13.

∴ The seat numbered 369 is located in the 13th row.

12. **(a)** **(i)** Let A_n cm² be the area of F_n.

∵ A_1, A_2, A_3, ... form an arithmetic sequence with first term 32 and common difference 8.5.

$$\therefore \quad A_n = 32 + (n-1)(8.5)$$

$$= 23.5 + 8.5n$$

∴ The area of F_{25}

$= A_{25}$ cm²

$= [23.5 + 8.5(25)]$ cm²

$= \underline{\underline{236 \text{ cm}^2}}$

(ii) The sum of areas of 25 figures

$= S(25)$ cm²

$= \frac{25}{2}(32 + 236)$ cm²

$= \underline{\underline{3350 \text{ cm}^2}}$

(b) **(i)** Let P_n cm be the perimeter of F_n.

∵ $F_1, F_2, F_3, \ldots, F_{25}$ are similar figures.

$$\left(\frac{P_2}{P_1}\right)^2 = \frac{A_2}{A_1}$$

$$\therefore \quad \left(\frac{36}{P_1}\right)^2 = \frac{32+8.5}{32}$$

$$\frac{36^2}{P_1^2} = \frac{81}{64}$$

$$P_1^2 = 1024$$

$$P_1 = 32 \text{ or } P_1 = -32 \text{ (rejected)}$$

∴ The perimeter of F_1 is 32 cm.

(ii)

$$\left(\frac{P_3}{P_2}\right)^2 = \frac{A_3}{A_2}$$

$$\left(\frac{P_3}{36}\right)^2 = \frac{32+2(8.5)}{32+8.5}$$

$$\frac{P_3^2}{36^2} = \frac{98}{81}$$

$$P_3^2 = 1568$$

$$P_3 = 28\sqrt{2} \text{ or } P_3 = -28\sqrt{2} \text{ (rejected)}$$

$$P_3 - P_2 = 28\sqrt{2} - 36 = 3.5979...$$

$$P_2 - P_1 = 36 - 32 = 4 \neq P_3 - P_2$$

∴ The perimeters of $F_1, F_2, F_3, \dots, F_{25}$ do not form an arithmetic sequence.

∴ Anthony's claim is not correct.

13. (a) (i) Total value at the end of the 1st year

$$= \$x(1+4\%) = \underline{\underline{\$1.04x}}$$

(ii) Total value at the end of 2nd year

$$= \$[x(1+4\%) + x(1+4\%)^2]$$
$$= \$(1.04x + 1.0816x)$$
$$= \underline{\underline{\$2.1216x}}$$

(b) Her investment at the end of each year forms a geometric sequence with first term \$1.04x and common ratio 1.04.

Total value at the end of the nth year

$$= \$\frac{x(1.04)(1.04^n - 1)}{1.04 - 1}$$
$$= \$\frac{x(1.04)(1.04^n - 1)}{0.04}$$
$$= \$26x(1.04^n - 1)$$

(c) Total value at the end of the 6th year

$$= \$26(20\ 000)(1.04^6 - 1) \quad \text{(from (b))}$$
$$= \underline{\underline{\$137\ 966}} \quad \text{(cor. to the nearest dollar)}$$

14. (a) (i) The distances that the train travels in each successive second form a geometric sequence with first term 20 m and common ratio = 80% = 0.8.

The distance travelled in the nth second

$$= \underline{\underline{20(0.8)^{n-1}\text{ m}}}$$

(ii) The total distance travelled in the first n seconds

$$= \frac{20(1-0.8^n)}{1-0.8}\text{ m}$$
$$= \frac{20(1-0.8^n)}{0.2}\text{ m}$$
$$= \underline{\underline{100(1-0.8^n)\text{ m}}}$$

(b) The total distance travelled

$$= \frac{20}{1-0.8} \text{ m}$$
$$= \frac{20}{0.2} \text{ m}$$
$$= 100 \text{ m}$$
$$< 101 \text{ m}$$

∴ The train can stop without hitting the obstacle.

15. (a) (i) The amount accumulated at the end of the 6th month

$$= \$\left[x\left(1+\frac{6\%}{12}\right) + x\left(1+\frac{6\%}{12}\right)^2 + \dots + x\left(1+\frac{6\%}{12}\right)^6\right]$$
$$= \$\frac{x(1.005)(1.005^6 - 1)}{1.005 - 1}$$
$$= \$201x(1.005^6 - 1)$$

(ii) The amount accumulated at the end of the nth year

$$= \$\frac{x(1.005)(1.005^{12n} - 1)}{1.005 - 1}$$
$$= \$201x(1.005^{12n} - 1)$$

(b) When $x = 3000$ and $n = 5$,
the required amount accumulated

$$= \$201(3000)[1.005^{12(5)} - 1]$$
$$= \underline{\underline{\$210\,357}} \text{ (cor. to nearest dollar)}$$

(c) Suppose Peter needs to deposit \$$k$ every month into the bank to save \$500 000 in 10 years.

$$\$201k[1.005^{12(10)} - 1] \geq \$500\,000$$
$$k \geq \frac{500\,000}{201(1.005^{120} - 1)}$$
$$k \geq 3036 \text{ (cor. to nearest dollar)}$$

∴ Peter needs to deposit \$3036 every month into the bank to save \$500 000 in 10 years.

16. (a) (i) The portion that P gets the first time $= \underline{\underline{\frac{1}{4}}}$

(ii) The portion that P gets the second time alone

$$= \left(\frac{1}{4}\right)\left(\frac{1}{4}\right) = \underline{\underline{\frac{1}{16}}}$$

(iii) The portion that P gets the nth time alone

$$= \left(\frac{1}{4}\right)^{n-1}\left(\frac{1}{4}\right)$$
$$= \underline{\underline{\frac{1}{4^n}}}$$

(b) The portion that P will get in the first n times

$$= \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \dots + \frac{1}{4^n}$$
$$= \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \dots + \frac{1}{4^n}$$

∴ The portion that P gets in each time forms a geometric sequence with first term $\frac{1}{4}$ and common ratio $\frac{1}{4}$.

∴ The portion that P will get if they divide the cake an infinite number of times

$$=\frac{\frac{1}{4}}{1-\frac{1}{4}}$$

$$=\frac{1}{3}$$

17. (a)

$$BD = AB\sin\theta$$

$$\therefore\quad d_1 = \underline{\underline{x\sin\theta}}$$

$$DE = BD\cos\angle EDB$$

$$\therefore\quad d_2 = \underline{\underline{x\sin\theta\cos\theta}}$$

(b) (i)

$$d_3 = EF = DE\cos\angle FED = x\sin\theta\cos^2\theta$$

$$d_4 = FG = EF\cos\angle GFE = x\sin\theta\cos^3\theta$$

$$\frac{d_2}{d_1} = \frac{x\sin\theta\cos\theta}{x\sin\theta} = \cos\theta$$

$$\frac{d_3}{d_2} = \frac{x\sin\theta\cos^2\theta}{x\sin\theta\cos\theta} = \cos\theta$$

$$\frac{d_4}{d_3} = \frac{x\sin\theta\cos^3\theta}{x\sin\theta\cos^2\theta} = \cos\theta$$

$$\because\quad \frac{d_2}{d_1} = \frac{d_3}{d_2} = \frac{d_4}{d_3} = \cos\theta$$

∴ d_1, d_2, d_3, d_4 form a geometric sequence with common ratio $\cos\theta$.

(ii)

$$d_1 + d_2 + d_3 + d_4 = \frac{x\sin\theta(1-\cos^4\theta)}{1-\cos\theta}$$

$$= \frac{x\sin\theta(1-\cos^2\theta)(1+\cos^2\theta)}{1-\cos\theta}$$

$$= \underline{\underline{x\sin\theta(1+\cos\theta)(1+\cos^2\theta)}}$$

(c)

$$d_1 + d_2 + d_3 + d_4$$

$$= 20\sin 30^\circ(1+\cos 30^\circ)(1+\cos^2 30^\circ)$$

$$= 20\left(\frac{1}{2}\right)\left(1+\frac{\sqrt{3}}{2}\right)\left[1+\left(\frac{\sqrt{3}}{2}\right)^2\right]$$

$$= \frac{35}{2}\left(\frac{2+\sqrt{3}}{2}\right)$$

$$= \underline{\underline{\frac{35}{4}(2+\sqrt{3})}}$$

18. (a) The required total revenue

$$= \$\frac{8[(1+5\%)^{n+1}-1]}{(1+5\%)-1}\text{ million}$$

$$= \underline{\underline{\$160(1.05^{n+1}-1)\text{ million}}}$$

(b) (i) The required total operation cost

$$= \$\frac{(n+1)}{2}[2(6)+n(0.5)]\text{ million}$$

$$= \underline{\underline{\$\frac{(n+1)(n+24)}{4}\text{ million}}}$$

(ii) The total revenue from 2013 to 2025

$$= \$160(1.05^{12+1}-1)\text{ million}$$

$$\approx \$141.7039\text{ million}$$

The total operation cost from 2013 to 2025

$$= \$\frac{(12+1)(12+24)}{4}\text{ million}$$

$$= \$117\text{ million}$$

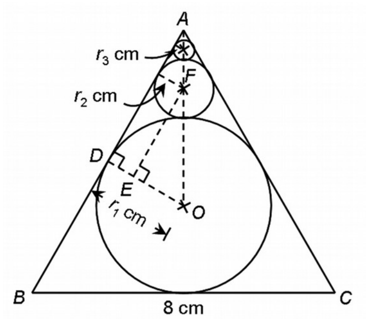
$$\$117\text{ million}\times 1.5 = \$175.5\text{ million}$$

Total revenue from 2013 to 2025

$\not\geq$ 1.5 × total operation cost from 2013 to 2025

∴ The special bonus will not be given in 2025.

19. (a)



∵ △ABC is an equilateral triangle.

∴ $\angle BAC = 60°$

Consider △ODA.

$$OD = AD\tan\angle OAD$$

$$r_1 = \frac{8}{2}\tan\frac{60^\circ}{2}$$

$$= 4\tan 30^\circ$$

$$= \underline{\underline{\frac{4}{\sqrt{3}}}}$$

Consider △OEF.

$$\angle FOE = 60^\circ$$

$$OE = OF\cos\angle FOE$$

$$r_1 - r_2 = (r_1 + r_2)\cos 60^\circ$$

$$r_2 = \frac{1}{3}r_1$$

$$= \underline{\underline{\frac{4}{3\sqrt{3}}}}$$

Similarly, $r_3 = \frac{1}{3}r_2 = \underline{\underline{\frac{4}{9\sqrt{3}}}}$

(b) (i) From (a), we know that r_1, r_2, r_3, … form a geometric sequence with first term $\frac{4}{\sqrt{3}}$ and common ratio $\frac{1}{3}$.

Sum of the circumferences of these circles

$=(2\pi r_1 + 2\pi r_2 + 2\pi r_3 + \dots)$ cm

$=2\pi(r_1 + r_2 + r_3 + \dots)$ cm

$=2\pi\left(\dfrac{\frac{4}{\sqrt{3}}}{1-\frac{1}{3}}\right)$ cm

$=\underline{\underline{4\sqrt{3}\pi \text{ cm}}}$

(ii) Consider the sequence $r_1^2, r_2^2, r_3^2, \dots$

$$\frac{r_n^2}{r_{n-1}^2} = \left(\frac{r_n}{r_{n-1}}\right)^2 = \frac{1}{9}$$

∴ $r_1^2, r_2^2, r_3^2, \dots$ form a geometric sequence with first term $=\left(\dfrac{4}{\sqrt{3}}\right)^2 = \dfrac{16}{3}$ and common ratio $\dfrac{1}{9}$.

Sum of the areas of these circles

$=(\pi r_1^2 + \pi r_2^2 + \pi r_3^2 + \dots)$ cm^2

$=\pi(r_1^2 + r_2^2 + r_3^2 + \dots)$ cm^2

$=\pi\left(\dfrac{\frac{16}{3}}{1-\frac{1}{9}}\right)$ cm^2

$=\underline{\underline{6\pi \text{ cm}^2}}$

Check Yourself (p. 2.42)

1. **(a)** ✘ **(b)** ✘

(c) ✓ **(d)** ✘

2. Suppose there are altogether k terms in the sequence.

$$\frac{k}{2}[8 + (-31)] = -161$$
$$k = 14$$

∴ There are altogether 14 terms in the sequence.

3. 1, 5, 9, ... is an arithmetic sequence with first term 1 and common difference $= 5 - 1 = 4$.

$$S(12) = \frac{12}{2}[2(1) + (12-1)(4)]$$
$$= \underline{\underline{276}}$$

4. 2, −4, 8, ... , −1024 is a geometric sequence with first term 2 and common ratio $= \dfrac{-4}{2} = -2$.

Suppose $T(k) = -1024$.

$$2(-2)^{k-1} = -1024$$
$$(-2)^{k-1} = (-2)^9$$
$$k - 1 = 9$$
$$k = 10$$
$$S(10) = \frac{2[1-(-2)^{10}]}{1-(-2)}$$
$$= \underline{\underline{-682}}$$

5. First term = 125, common ratio $= \dfrac{75}{125} = \dfrac{3}{5}$

$$S(\infty) = \frac{125}{1-\frac{3}{5}}$$
$$= \underline{\underline{\frac{625}{2}}}$$

6. **(a)** Let a and r be the first term and the common ratio of the sequence respectively.

$$T(3) = ar^2 = 6 \quad \text{……(1)}$$

$$T(6) = ar^5 = \frac{2}{9} \quad \text{……(2)}$$

$$\frac{(2)}{(1)}: \quad r^3 = \frac{\frac{2}{9}}{6}$$

$$r^3 = \frac{1}{27}$$

$$r = \frac{1}{3}$$

By substituting $r = \frac{1}{3}$ into (1), we have

$$a\left(\frac{1}{3}\right)^2 = 6$$

$$a = 54$$

∴ The first term is 54 and the common ratio is $\frac{1}{3}$.

(b)

$$S(5) = \frac{54\left[1 - \left(\frac{1}{3}\right)^5\right]}{1 - \frac{1}{3}}$$

$$= \underline{\underline{\frac{242}{3}}}$$

7. The multiples of 6 between 100 and 200 inclusive are:

102, 108, 114, … , 198

They form an arithmetic sequence with first term 102 and common difference 6.

Let m be the number of terms in 102, 108, 114, … , 198.

$$198 = 102 + (m - 1)(6)$$

$$m = 17$$

∴ The sum of all the multiples of 6 between 100 and 200 inclusive

$$= \frac{17}{2}(102 + 198)$$

$$= \underline{\underline{2550}}$$

8. Suppose the total production of beer will first exceed 200 000 L in the kth month starting from January (i.e. the 1st month is January).

$$\frac{20\,000[(1+2\%)^k - 1]}{(1+2\%) - 1} > 200\,000$$

$$1.02^k - 1 > 0.2$$

$$1.02^k > 1.2$$

$$\log 1.02^k > \log 1.2$$

$$k \log 1.02 > \log 1.2$$

$$k > \frac{\log 1.2}{\log 1.02}$$

$$> 9.2069...$$

Since k is an integer, the minimum value of k is 10.

∴ The total production of beer will first exceed 200 000 L

in the 10th month.
i.e. The total production of beer will first exceed 200 000 L in October.

Revision Exercise 2 (p. 2.43)
Level 1

1. Let a and d be the first term and the common difference of the sequence respectively.

(a) $\because$ $a = -3$ and $d = 2 - (-3) = 5$

$$\therefore \quad S(20) = \frac{20}{2}[2(-3) + (20-1)(5)] = \underline{\underline{890}}$$

(b) $\because$ $a = 65$ and $d = 62 - 65 = -3$

$$\therefore \quad S(15) = \frac{15}{2}[2(65) + (15-1)(-3)] = \underline{\underline{660}}$$

(c) $\because$ $a = 7$ and $d = 4 - 7 = -3$

Let n be the number of terms in the sequence.

$$-26 = 7 + (n-1)(-3)$$
$$n = 12$$
$$\therefore \quad S(12) = \frac{12}{2}[7 + (-26)] = \underline{\underline{-114}}$$

2. Let a and r be the first term and the common ratio of the sequence respectively.

(a) $\because$ $a = 3$ and $r = \frac{6}{3} = 2$

$$\therefore \quad S(10) = \frac{3(2^{10}-1)}{2-1} = \underline{\underline{3069}}$$

(b) $\because$ $a = 8$ and $r = \frac{-24}{8} = -3$

$$\therefore \quad S(8) = \frac{8[1-(-3)^8]}{1-(-3)} = \frac{8(1-6561)}{4} = \underline{\underline{-13120}}$$

(c) $\because$ $a = -18$ and $r = \frac{-6}{-18} = \frac{1}{3}$

Let k be the number of terms in the sequence.

$$-18\left(\frac{1}{3}\right)^{k-1} = -\frac{2}{27}$$
$$\left(\frac{1}{3}\right)^{k-1} = \frac{1}{243}$$
$$\left(\frac{1}{3}\right)^{k-1} = \left(\frac{1}{3}\right)^5$$
$$k - 1 = 5$$
$$k = 6$$

$$\therefore \quad S(6) = \frac{-18\left[1-\left(\frac{1}{3}\right)^6\right]}{1-\frac{1}{3}}$$
$$= -\frac{728}{27}$$

3. Let a and r be the first term and the common ratio of the sequence respectively.

(a) $\because \quad a = 28$ and $r = \frac{14}{28} = \frac{1}{2}$

$$\therefore \quad S(\infty) = \frac{28}{1-\frac{1}{2}}$$
$$= 56$$

(b) $\because \quad a = 27$ and $r = \frac{-18}{27} = -\frac{2}{3}$

$$\therefore \quad S(\infty) = \frac{27}{1-\left(-\frac{2}{3}\right)}$$
$$= \frac{81}{5}$$

(c) $\because \quad a = -\frac{3}{5}$ and $r = \frac{-\frac{9}{25}}{-\frac{3}{5}} = \frac{3}{5}$

$$\therefore \quad S(\infty) = \frac{-\frac{3}{5}}{1-\frac{3}{5}}$$
$$= -\frac{3}{2}$$

4. **(a)** Let k be the number of terms in the sequence.

$\because \quad S(k) = 10\,100$

$$\therefore \quad \frac{k}{2}(200+2) = 10\,100$$
$$101k = 10\,100$$
$$k = 100$$

$\therefore$ There are 100 terms in the sequence.

(b) Let d be the common difference.

$\because \quad T(100) = 2$

$$\therefore \quad 200 + (100-1)d = 2$$
$$99d = -198$$
$$d = -2$$

$\therefore$ The common difference is –2.

5. **(a)** Let a and d be the first term and the common difference of the sequence respectively.

$\because \quad S(9) = 90$

$$\therefore \quad \frac{9}{2}[2a + (9-1)d] = 90$$
$$a + 4d = 10 \quad \text{……(1)}$$

$\because \quad T(11) = 14$

$\therefore \quad a + 10d = 14 \quad$ ……(2)

(2) − (1):
$$6d = 4$$
$$d = \frac{2}{3}$$

∴ The common difference is $\frac{2}{3}$.

(b) By substituting $d = \frac{2}{3}$ into (1), we have

$$a + 4\left(\frac{2}{3}\right) = 10$$
$$a = \frac{22}{3}$$

∴
$$S(50) = \frac{50}{2}\left[2\left(\frac{22}{3}\right) + (50-1)\left(\frac{2}{3}\right)\right]$$
$$= \frac{3550}{3}$$

6. **(a)** The multiples of 7 between 100 and 400 inclusive are:
105, 112, 119, … , 399
They form an arithmetic sequence with first term 105 and common difference 7.
Let *m* be the number of terms in 105, 112, 119, ... , 399.
$$399 = 105 + (m-1)(7)$$
$$m = 43$$
∴ There are 43 multiples of 7 between 100 and 400 inclusive.

(b) The sum of all the multiples of 7 between 100 and 400 inclusive
$$= \frac{43}{2}(105 + 399)$$
$$= 10\,836$$

7. Let *a* and *r* be the first term and the common ratio of the sequence respectively.

∵ $a = 1$ and $r = \frac{3}{1} = 3$

∴
$$S(k) > 2000$$
$$\frac{1(3^k - 1)}{3-1} > 2000$$
$$3^k - 1 > 4000$$
$$\log 3^k > \log 4001$$
$$k \log 3 > \log 4001$$
$$k > \frac{\log 4001}{\log 3}$$
$$k > 7.5497...$$

Since *k* is an integer, the minimum value of *k* is 8.

8. Let *a* and *d* be the first term and the common difference of the sequence respectively.

∵ $a = \frac{2}{3}$ and $d = \frac{5}{3} - \frac{2}{3} = 1$

∴
$$S(k) < 3650$$
$$\frac{k}{2}\left[2\left(\frac{2}{3}\right) + (k-1)(1)\right] < 3650$$
$$k\left(\frac{1}{3} + k\right) < 7300$$
$$k(1+3k) < 21900$$
$$3k^2 + k - 21900 < 0$$
$$-85.6068... < k < 85.2735...$$

∴ The maximum value of k is 85.

9. **(a)** Let r be the common ratio of the sequence.
First term = 66

$$\because \quad S(2) = 88$$
$$66 + 66r = 88$$
$$\therefore \quad 3r = 1$$
$$r = \frac{1}{3}$$
$$\because \quad -1 < r < 1$$

∴ The sum to infinity of the sequence exists.

(b)
$$S(\infty) = \frac{66}{1 - \frac{1}{3}}$$
$$= \underline{\underline{99}}$$

10. **(a)** First term = a

Common difference $= (3a + b) - a$
$= 2a + b$

$$\therefore \quad S(10) = \frac{10}{2}[2(a) + (10-1)(2a+b)]$$
$$= 5(20a + 9b)$$
$$= \underline{\underline{100a + 45b}}$$

(b) Consider $a = 100$ and $b = 10$.

$$T(1) = 100 = 100$$
$$T(2) = 310 = 3(100) + 10$$
$$T(3) = 520 = 5(100) + 2(10)$$
$$T(4) = 730 = 7(100) + 3(10)$$
$$\vdots$$

∴ The sum of the first 10 terms
$$= 100(100) + 45(10) \quad \text{(from (a))}$$
$$= \underline{\underline{10\ 450}}$$

11. **(a)** ∵ First term = 9 and common ratio $= \frac{3}{9} = \frac{1}{3}$

∴ The absolute error in her answer
$$= S(\infty) - S(7)$$
$$= \frac{9}{1 - \frac{1}{3}} - \frac{9\left[1 - \left(\frac{1}{3}\right)^7\right]}{1 - \frac{1}{3}}$$
$$= \frac{27}{2} - \frac{1093}{81}$$
$$= \underline{\underline{\frac{1}{162}}}$$

(b) Percentage error
$$= \frac{\frac{1}{162}}{\frac{27}{2}} \times 100\%$$
$$= \underline{\underline{0.0457\%}} \text{ (cor. to 3 sig. fig.)}$$

12. **(a)** Let a and d be the first term and the common difference of the sequence respectively.

$$S(10)=350$$
$$\frac{10}{2}[2a+(10-1)d]=350$$
$$2a+9d=70 \quad \ldots\ldots(1)$$
$$S(20)-S(10)=950$$
$$\frac{20}{2}[2a+(20-1)d]-350=950$$
$$20a+190d=1300 \quad \ldots\ldots(2)$$
$$2a+19d=130$$

(2) – (1): $10d=60$
$$d=6$$

By substituting $d=6$ into (1), we have
$$2a+9(6)=70$$
$$a=8$$

$$\therefore \quad S(15)=\frac{15}{2}[2(8)+(15-1)(6)]$$
$$=\underline{\underline{750}}$$

(b) The sum from the 20th term to the 30th term

$$=S(30)-S(19)$$
$$=\frac{30}{2}[2(8)+(30-1)(6)]-\frac{19}{2}[2(8)+(19$$
$$=2850-1178$$
$$=\underline{\underline{1672}}$$

13. **(a)** First term = 1 and common ratio $=\frac{3}{1}=3$

Suppose $T(k)$ = 729.
$$1(3)^{k-1}=729$$
$$\therefore \quad 3^{k-1}=729$$
$$3^{k-1}=3^6$$
$$k-1=6$$
$$k=7$$
$$\therefore \quad S(7)=\frac{1(3^7-1)}{3-1}$$
$$=\frac{2187-1}{2}$$
$$=\underline{\underline{1093}}$$

(b)
$$2\times2^3\times2^9\times\ldots\times2^{729}=4^x$$
$$2^{1+3+9+\ldots+729}=(2^2)^x$$
$$2^{1093}=2^{2x}$$
$$2x=1093 \quad \text{(from (a))}$$
$$x=\underline{\underline{\frac{1093}{2}}}$$

14. Let a and r be the first term and the common ratio of the sequence respectively.

$$\because \quad S(\infty)=\frac{7}{9}a$$
$$\frac{a}{1-r}=\frac{7}{9}a$$
$$\therefore \quad \frac{1}{1-r}=\frac{7}{9}$$
$$1-r=\frac{9}{7}$$
$$r=-\frac{2}{7}$$

$\therefore$ The common ratio of the sequence is $-\frac{2}{7}$.

15. **(a)** First term = 6 and common ratio $=\frac{-\frac{18}{5}}{6}=-\frac{3}{5}$

$$\therefore \quad S(\infty)=\frac{6}{1-\left(-\frac{3}{5}\right)}$$
$$=\underline{\underline{\frac{15}{4}}}$$

(b) **(i)** Let $T_1(n)$ be the nth term of the sequence
$$6,\ -\frac{18}{5},\ \frac{54}{25},\ -\frac{162}{125},\ \ldots$$
Let $T_2(n)$ be the nth negative term of the sequence.
$$\frac{T_2(n)}{T_2(n-1)}=\frac{T_1(2n)}{T_1(2n-2)}$$
$$=\frac{6\left(-\frac{3}{5}\right)^{2n-1}}{6\left(-\frac{3}{5}\right)^{2n-3}}$$
$$=\left(-\frac{3}{5}\right)^2$$
$$=\frac{9}{25}, \text{ which is a constant.}$$

$\therefore$ The negative terms of the sequence form a geometric sequence.
$\therefore$ Eddie's claim is correct.

(ii) The sum of all the negative terms of the sequence
$$=\frac{-\frac{18}{5}}{1-\frac{9}{25}}$$
$$=\frac{-\frac{18}{5}}{\frac{16}{25}}$$
$$=\underline{\underline{-\frac{45}{8}}}$$

16. Let r be the common ratio of the sequence formed.

$\because$ First term = 6

$T(6) = -6144$

and $6r^5 = -6144$

$r^5 = -1024$

$r = -4$

$\therefore$ The sum of the 4 numbers inserted

$$= S(5) - T(1)$$
$$= \frac{6[1-(-4)^5]}{1-(-4)} - 6$$
$$= 1230 - 6$$
$$= \underline{\underline{1224}}$$

17. The time David spent on reading in successive days forms an arithmetic sequence with first term 20 minutes and common difference 4 minutes.

$\therefore$ Total time that David spent on reading in the first 3 weeks (i.e. 21 days) of the holiday

$$= \frac{21}{2}[2(20) + (21-1)(4)] \text{ minutes}$$
$$= \underline{\underline{1260 \text{ minutes}}}$$

18. The lengths of successive parts of the string form an arithmetic sequence with first term (the longest part) a cm and common difference –1 cm.

$\because$ $S(26) = 429$

$\therefore$ $\frac{26}{2}[2a + (26-1)(-1)] = 429$

$2a - 25 = 33$

$a = 29$

$\therefore$ The length of the longest part $= \underline{\underline{29 \text{ cm}}}$

$\therefore$ The length of the shortest part

$$= T(26)$$
$$= [29 + (26-1)(-1)] \text{ cm}$$
$$= \underline{\underline{4 \text{ cm}}}$$

19. The productions of steel in successive months form a geometric sequence with first term 35 000 tonnes and common ratio = (1 + 5%) = 1.05.

$\therefore$ Total production of steel in the first half year

$$= \frac{35\,000(1.05^6 - 1)}{1.05 - 1} \text{ tonnes}$$
$$= \underline{\underline{238\,000 \text{ tonnes}}} \text{ (cor. to the nearest thousand)}$$

20. The numbers of visitors in successive months from January 2014 form a geometric sequence with first term 120 000 and common ratio = (1 + 7%) = 1.07.

Suppose the total number of visitors will first exceed 1 000 000 in the kth month starting from January 2014.

$$\frac{120\,000(1.07^k - 1)}{1.07 - 1} > 1\,000\,000$$
$$1.07^k > \frac{19}{12}$$
$$\log(1.07^k) > \log\frac{19}{12}$$
$$k > \frac{\log\frac{19}{12}}{\log 1.07}$$
$$k > 6.7919...$$

The total number of visitors will first exceed 1 000 000 in July 2014.

$\therefore$ The lucky draw will be launched in August 2014.

21. **(a)** The maximum heights the ball can reach in successive rebounds form a geometric sequence with first term = 3(80%) m = 2.4 m and common ratio = 80% = 0.8.

$\therefore$ Total distance travelled by the ball before the 4th rebound

$$= \left[3 + 2 \times \frac{2.4(1-0.8^3)}{1-0.8}\right] \text{ m}$$
$$= \underline{\underline{14.712 \text{ m}}}$$

(b) Total distance travelled by the ball before it comes to rest

$$=\left(3+2\times\frac{2.4}{1-0.8}\right)\text{ m}$$
$$=\underline{\underline{27\text{ m}}}$$

22. (a) (i) The numbers of seats in successive rows form an arithmetic sequence with first term 12 and common difference 3.

Let $T(k)$ be the number of seats in the last row.

$\because$ $S(k)=810$

$$\frac{k}{2}[2(12)+(k-1)(3)]=810$$

$\therefore$
$$3k^2+21k=1620$$
$$k+7k-540=0$$
$$(k-20)(k+27)=0$$
$$k=20\text{ or }k=-27\text{(rejected)}$$

$\therefore$ There are 20 rows of seats.

(ii) Total number of seats in the first 9 rows

$$=\frac{9}{2}[2(12)+(9-1)(3)]$$
$$=216$$

$\therefore$ The smallest seat number in the 10th row is 216 + 1 = 217.

(b) Suppose the seat numbered 500 is located in the mth row.

$$S(m)\geq 500$$
$$\frac{m}{2}[2(12)+(m-1)(3)]\geq 500$$
$$3m^2+21m\geq 1000$$
$$3m^2+21m-1000\geq 0$$

$\therefore$ $m\geq\dfrac{-21+\sqrt{21^2-4(3)(-1000)}}{2(3)}$

or

$m\leq\dfrac{-21-\sqrt{21^2-4(3)(-1000)}}{2(3)}$ (rejected)

$\therefore$ $m\geq 15.0898...$

Since m is an integer, the minimum value of m is 16.

$\therefore$ Vivian's seat is located in the 16th row.

23. (a) The radii of successive circles form a geometric sequence with first term 10 cm and common ratio = 1 + 20% = 1.2.

Let r_n cm be the radius of the nth circle.

$\therefore$ $r_n=10(1.2)^{n-1}$

$\therefore$ The sum of the circumference of the first 4 circles

$$=2r_1\pi+2r_2\pi+2r_3\pi+2r_4\pi$$
$$=2\pi(r_1+r_2+r_3+r_4)$$
$$=2\pi\times\frac{10(1.2^4-1)}{1.2-1}\text{ cm}$$
$$=\underline{\underline{337\text{ cm}}}\text{ (cor. to the nearest integer)}$$

(b) $\because$ Each circles are similar.

$\therefore$ The areas of successive circles form a geometric sequence with first term = $10^2\pi\text{ cm}^2=100\pi\text{ cm}^2$ and common ratio = $(1+20\%)^2=1.44$.

Let A_n cm^2 be the area of the nth circle.

$\therefore$ $A_n=100(1.44)^{n-1}\pi$

∴ The sum of the areas of the first 8 circles

$$= A_1 + A_2 + \dots + A_8$$
$$= \frac{100\pi(1.44^8 - 1)}{1.44 - 1} \text{ cm}^2$$
$$= \underline{\underline{12\ 487 \text{ cm}^2}} \text{ (cor. to the nearest integ}$$

Level 2

24. (a) ∵ First term = –71

and common difference $= -65 - (-71) = 6$

∴ $T(n) = -71 + (n-1)(6) = 6n - 77$

Suppose $T(k)$ is the greatest negative number in the sequence.

$$T(k) < 0$$
∴
$$6k - 77 < 0$$
$$6k < 77$$
$$k < 12.8333...$$

∴ There is 12 negative terms in the sequence.

∴ The sum of all the negative terms in the sequence

$$= \frac{12}{2}[2(-71) + (12-1)(6)]$$
$$= \underline{\underline{-456}}$$

(b) Suppose $T(m) = 61$.

$$6m - 77 = 61$$
$$m = 23$$

∴ The sum of all the positive terms in the sequence

$$= \frac{23}{2}(-71 + 61) - (-456)$$
$$= \underline{\underline{341}}$$

25. (a) ∵ $S(n) = 16n - n^2$

$T(n) = S(n) - S(n-1)$

∴
$$= (16n - n^2) - [16(n-1) - (n-1)^2]$$
$$= 16n - n^2 - (16n - 16 - n^2 + 2n - 1)$$
$$= \underline{\underline{17 - 2n}}$$

(b) ∵

$$T(n) - T(n-1) = 17 - 2n - [17 - 2(n-1)]$$
$$= -2, \text{ which is a constant.}$$

∴ The sequence is an arithmetic sequence.

26. (a)

$$a \times a^2 \times a^3 \times \dots \times a^{10}$$
$$= a^{1+2+3+\dots+10}$$
$$= a^{\frac{10}{2}(1+10)}$$
$$= \underline{\underline{a^{55}}}$$

(b)

$$\log 2 + \log 4 + \log 8 + \dots \text{ to } 10 \text{ terms}$$
$$= \log 2 + \log 2^2 + \log 2^3 + \dots + \log 2^{10}$$
$$= \log(2 \times 2^2 \times 2^3 \times \dots \times 2^{10})$$
$$= \log 2^{55} \text{ (from (a))}$$
$$= \underline{\underline{55 \log 2}}$$

27. (a) Let $T(n)$ be the nth term of the sequence.

$T(1) = \log a$

$T(2) = \log 10a = \log a + 1$

$T(3) = \log 100a = \log a + \log 10^2 = \log a + 2$

$\vdots$

$T(n) = \log 10^{n-1}a = \log a + n - 1$

$\therefore \quad T(n) - T(n-1)$
$= \log a + n - 1 - (\log a + n - 2)$
$= 1$, which is a constant.

$\therefore$ The sequence is an arithmetic sequence.

(b) The sum of the first 10 terms

$$= S(10)$$
$$= \frac{10}{2}[2 \log a + (10 - 1)(1)]$$
$$= \underline{\underline{10 \log a + 45}}$$

28. (a)

$$2T(1) + 2T(2) + 2T(3) + \dots$$
$$= 2[T(1) + T(2) + T(3) + \dots]$$
$$= 2(6)$$
$$= \underline{\underline{12}}$$

(b) Let *a* and *r* be the first term and the common ratio of the sequence respectively.

$$S(\infty) = \frac{a}{1-r} = 6 \quad \text{......(1)}$$

$$2T(1) + 2T(2) + 2T(3) = \frac{21}{2}$$

$$\frac{a(1-r^3)}{1-r} = \frac{21}{4} \quad \text{......(2)}$$

$$\frac{\frac{a(1-r^3)}{1-r}}{\frac{a}{1-r}} = \frac{\frac{21}{4}}{6}$$

$$\frac{(2)}{(1)}: \quad 1 - r^3 = \frac{7}{8}$$

$$r^3 = \frac{1}{8}$$

$$r = \frac{1}{2}$$

By substituting $r = \frac{1}{2}$ into (1), we have

$$\frac{a}{1-\frac{1}{2}} = 6$$

$$a = 3$$

$\therefore \quad T(1) = a = \underline{\underline{3}}$

29. John's saving at the end of each year from 2013 forms a geometric sequence with first term = \$156 000(20%) = \$31 200 and common ratio = (1 + 5%) = 1.05.

$\therefore$ John's total savings at the end of 2023

$$= \$\frac{31\ 200(1.05^{11} - 1)}{1.05 - 1}$$
$$= \underline{\underline{\$443\ 000}} \text{ (cor. to the nearest \$1000)}$$

30. The lengths of the pile driven into the ground in successive blows form a geometric sequence with first term 2 m and common ratio 0.9.

(a) The depth after 6 blows

$= S(6)$

$= \dfrac{2(1-0.9^6)}{1-0.9}$ m

$= \dfrac{2(1-0.9^6)}{0.1}$ m

$= \underline{\underline{9.37 \text{ m}}}$ (cor. to 3 sig. fig.)

(b) The depth after 7 blows

$= S(7)$

$= \dfrac{2(1-0.9^7)}{1-0.9}$ m

$= \dfrac{2(1-0.9^7)}{0.1}$ m

$= 10.4340...$ m

> 10 m

∴ The pile will be completely driven into the ground with one more blow.

31. (a) The height increases of the plant in successive months form a geometric sequence with first term 2 cm and common ratio = 80% = 0.8.

∴ The height increase of the plant in the *n*th month

$= \underline{\underline{2(0.8)^{n-1} \text{ cm}}}$

(b) The total height increase

$= \dfrac{2}{1-0.8}$ cm

$= \dfrac{2}{0.2}$ cm

$= 10$ cm

∴ The height of the plant after a long period

$= (80+10)$ cm

$= \underline{\underline{90 \text{ cm}}}$

32. The diameters of successive semi-circles form a geometric sequence with first term 8 mm and common ratio = 60% = 0.6.

∴ The lengths of successive semi-circles form a geometric sequence with first term =

$\dfrac{1}{2}(8\pi)$ mm $= 4\pi$ mm and common ratio 0.6.

∴ Maximum length of the spiral curl of the snail shell

$= \dfrac{4\pi}{1-0.6}$ mm

$= \dfrac{4\pi}{0.4}$ mm

$= \underline{\underline{10\pi \text{ mm}}}$

33. (a) The interior angles of polygon form an arithmetic sequence with first term 132° and common difference –12°.

Suppose the polygon has *k* sides.

$S(k) = (k-2)\times 180^\circ$ (∠ sum of polygon)

$\dfrac{k}{2}[2(132)+(k-1)(-12)] = (k-2)180$

$-6k^2 + 138k = 180k - 360$

$6k^2 + 42k - 360 = 0$

$k^2 + 7k - 60 = 0$

$(k-5)(k+12) = 0$

$k = 5$ or $k = -12$ (rejected)

∴ The number of sides is 5.

(b) The second smallest interior angle

$= T(4)$
$= 132° + (4-1)(-12°)$
$= \underline{\underline{96°}}$

34. (a) $AC_1 = AC - C_1C = 3a - b$

$\because$ $\triangle AB_1C_1 \sim \triangle ABC$ (AAA)

$\therefore$ $\dfrac{AC_1}{AC} = \dfrac{B_1C_1}{BC}$ (corr. sides, ~△s)

$\dfrac{3a-b}{3a} = \dfrac{b}{a}$

$3a - b = 3b$

$3a = 4b$

$b = \underline{\underline{\dfrac{3}{4}a}}$

(b) (i) From (a), we have $B_1C_1 = \dfrac{3}{4}BC$.

Similarly, we have

$B_2C_2 = \dfrac{3}{4}B_1C_1$

$= \underline{\underline{\dfrac{3}{4}b}}$

(ii) $B_2C_2 = \dfrac{3}{4}b$

$= \dfrac{3}{4}\left(\dfrac{3}{4}a\right)$

$= \underline{\underline{\dfrac{9}{16}a}}$

(c) (i) $\because$ $B_1C_1 = \dfrac{3}{4}a$

$B_2C_2 = \dfrac{9}{16}a = \left(\dfrac{3}{4}\right)^2 a$

$B_3C_3 = \dfrac{3}{4}B_2C_2$

$= \dfrac{3}{4}\left(\dfrac{3}{4}\right)^2 a$

$= \left(\dfrac{3}{4}\right)^3 a$

$\therefore$ $B_nC_n = \left(\dfrac{3}{4}\right)^n a$

$\dfrac{B_nC_n}{B_{n-1}C_{n-1}} = \dfrac{\left(\frac{3}{4}\right)^n a}{\left(\frac{3}{4}\right)^{n-1} a}$

$= \dfrac{3}{4}$, which is a constant.

$\therefore$ B_1C_1, B_2C_2, B_3C_3, … form a geometric sequence.

(ii) $B_4C_4 = \left(\dfrac{3}{4}\right)^4 a$

$= \underline{\underline{\dfrac{81}{256}a}}$

(iii) The areas of the squares form a geometric sequence with first term $= \left(\dfrac{3}{4}a\right)^2 = \dfrac{9}{16}a^2$ and common ratio $= \left(\dfrac{3}{4}\right)^2 = \dfrac{9}{16}$.

$\therefore$ The required sum of areas

$= (B_1C_1)^2 + (B_2C_2)^2 + (B_3C_3)^2 + \dots$

$= \dfrac{\frac{9}{16}a^2}{1 - \frac{9}{16}}$

$= \underline{\underline{\dfrac{9}{7}a^2}}$

35. (a) $\because$ The speed of Ken is twice that of Angel.

$\therefore$ The distance travelled by Ken is twice that of Angel in the same time.

$\therefore$ $BC = \dfrac{1}{2}AB = \dfrac{1}{2}(24\text{ m}) = \underline{\underline{12\text{ m}}}$

Similarly, $CD = \dfrac{1}{2}BC = \dfrac{1}{2}(12\text{ m}) = \underline{\underline{6\text{ m}}}$

and $DE = \dfrac{1}{2}CD = \dfrac{1}{2}(6\text{ m}) = \underline{\underline{3\text{ m}}}$

(b) $\dfrac{BC}{AB} = \dfrac{12}{24} = \dfrac{1}{2}$

$\dfrac{CD}{BC} = \dfrac{6}{12} = \dfrac{1}{2}$

$\therefore$ AB, BC, CD, … form a geometric sequence with common ratio $\dfrac{1}{2}$.

(c) Total distance required

$= \dfrac{24}{1 - \frac{1}{2}}$ m

$= \underline{\underline{48\text{ m}}}$

(d) Only with $r > 1$, Ken can eventually catch up with Angel.

$\because$ The speed of Ken is r times that of Angel.

$\therefore$ The distance travelled by Ken is r times that of Angel in the same time.

$\therefore$ $\dfrac{BC}{AB} = \dfrac{1}{r}$ and $\dfrac{CD}{BC} = \dfrac{1}{r}$

$\therefore$ AB, BC, CD, … form a geometric sequence with common ratio $\dfrac{1}{r}$.

$\therefore$ Total distance Ken must run to meet Angel

$$= \frac{24}{1 - \frac{1}{r}} \text{ m}$$
$$= \frac{24}{\frac{r - 1}{r}} \text{ m}$$
$$= \frac{24r}{r - 1} \text{ m}$$

∴ They will meet at $\frac{24r}{r-1}$ m away from A.

36. **(a)** **(i)** Her annual salary in the 2nd year

$$= \$11\,000(1 + 9\%) \times 12$$
$$= \underline{\underline{\$143\,880}}$$

(ii) Her annual salary in the 5th year

$$= \$11\,000(1 + 9\%)^4 \times 12$$
$$= \underline{\underline{\$186\,329}} \text{ (cor. to the nearest dollar)}$$

(b) Total salary in the first five years in Company X

$$= \$\frac{11\,000(12)(1.09^5 - 1)}{1.09 - 1}$$
$$= \$789\,982 \text{ (cor. to the nearest dollar)}$$

Total salary in the first five years in Company Y

$$= \$\frac{5}{2}[2(12\,000)(12) + (5 - 1)(400 \times 12)]$$
$$= \$768\,000$$

∴ Carmen will join Company X for a higher total salary.

37. **(a)** **(i)** Let r and A_n cm^2 be the common ratio and the area of T_n respectively.

∵ $A_1 = 200$

∴

$$A_3 = 180.5$$
$$200r^2 = 180.5$$
$$r = 0.95 \quad \text{or} \quad r = -0.95 \text{(rejected)}$$

∴ Area of T_{10}

$$= A_{10} \text{ cm}^2$$
$$= 200(0.95)^9 \text{ cm}^2$$
$$= \underline{\underline{126 \text{ cm}^2}} \text{ (cor. to the nearest integer)}$$

(ii) The sum of the areas of T_1, T_2, T_3, ...

$$= (A_1 + A_2 + A_3 + \ldots) \text{ cm}^2$$
$$= \frac{200}{1 - 0.95} \text{ cm}^2$$
$$= \underline{\underline{4000 \text{ cm}^2}}$$

(b) ∵ V_1, V_2, V_3, ... are similar solid tetrahedrons.

∴

$$\sqrt[3]{\frac{\text{Volume of } V_n}{\text{Volume of } V_{n-1}}} = \sqrt{\frac{\text{Base area of } V_n}{\text{Base area of } V_{n-1}}}$$
$$\frac{\text{Volume of } V_n}{\text{Volume of } V_{n-1}} = (0.95)^{\frac{3}{2}}, \text{ which is a constant.}$$

∴ Volumes of V_1, V_2, V_3, ... form a geometric sequence.

∴ Vincent's claim is correct.

38. **(a)** The amounts accumulated at the end of each month in Bank A form a geometric sequence with first term $= \$6000\left(1 + \frac{6\%}{12}\right) = \$6030$ and common ratio $= 1 + \frac{6\%}{12} = 1.005$.

The amount accumulated after 1 year

$$=\$\frac{6030(1.005^{12}-1)}{1.005-1}$$
$$=\$74\ 383.4411...$$
$$>\$73\ 000$$

∴ The amount is enough to pay Darren's salaries tax.

(b) Suppose the total amount accumulated in both banks will be enough to pay his salaries tax in k months.
The amount accumulated in Bank A after k months

$$=\$\frac{6030(1.005^{k}-1)}{1.005-1}$$

The amounts accumulated at the end of each month in Bank B form a geometric sequence with first term

$$=\$2000\left(1+\frac{12.03\%}{12}\right)=\$2020.05$$ and

common ratio

$$=1+\frac{12.03\%}{12}=1.010\ 025=1.005^2.$$

The amount accumulated in Bank B after k months

$$=\$\frac{2020.05(1.005^{2k}-1)}{1.005^2-1}$$

$$\frac{6030(1.005^k-1)}{1.005-1}+\frac{2020.05(1.005^{2k}-1)}{1.005^2-1}>73\ 000$$
$$12\ 090.15(1.005^k-1)+2020.05(1.005^{2k}-1)>731.825$$
$$80\ 802(1.005^k)^2+483\ 606(1.005^k)-593\ 681>0$$
$$\therefore\quad 1.005^k<-7.03019\text{ (rejected)}$$
$$\text{or}\quad 1.005^k>1.045\ 115$$
$$\log 1.005^k>\log 1.045\ 115$$
$$k\log 1.005>\log 1.045\ 115$$
$$k>8.8474...$$

Since k is an integer, the minimum value of k is 9.
∴ The total amount accumulated in both banks will be enough to pay his salaries tax in 9 months.

39. (a)

$$600\ 000(1-r\%)^2=486\ 000$$
$$(1-r\%)^2=0.81$$
$$1-r\%=0.9\quad\text{or}\quad 1-r\%=-0.9\text{ (rejected)}$$
$$r=\underline{\underline{10}}$$

(b) (i) Suppose it takes k years for the total revenue exceed \$1 300 000.

$$\frac{300\ 000[1-(1-19\%)^k]}{1-(1-19\%)}>1\ 300\ 000$$
$$\frac{1-0.81^k}{0.19}>\frac{13}{3}$$
$$1-0.81^k>\frac{247}{300}$$
$$0.81^k<\frac{53}{300}$$
$$\log 0.81^k<\log\frac{53}{300}$$
$$k\log 0.81<\log\frac{53}{300}$$
$$k>\frac{\log\frac{53}{300}}{\log 0.81}$$
$$k>8.2264...$$

∴ It takes at least 9 years for the factory to make the total revenue more than \$1 300 000.

(ii) Total revenue after a long period of time

$$=\$\frac{300\ 000}{1-(1-19\%)}$$
$$\approx \$1\ 578\ 947.37$$
$$< \$1\ 600\ 000$$

∴ The total revenue will not exceed $1 600 000.

(iii) Suppose the factory will be reformed in the *m*th year since 2011.
Total production cost made in the *m* years

$$=\frac{\$600\ 000[1-(1-10\%)^m]}{1-(1-10\%)}$$
$$=\$6(1-0.9^m)\text{million}$$

Total revenue made in the *m* years

$$=\frac{\$300\ 000[1-(1-19\%)^m]}{1-(1-19\%)}$$
$$=\$\frac{30}{19}(1-0.81^m)\text{ million}$$

∴

$$6(1-0.9^m)-\frac{30}{19}(1-0.81^m)>1.2$$
$$190(1-0.9^m)-50(1-0.9^{2m})>38$$
$$-190(0.9^m)+50(0.9^m)^2>-102$$
$$25(0.9^m)^2-95(0.9^m)+51>0$$

∴ $0.9^m > 3.1530$ (rejected)

$$0.9^m < 0.647\ 004$$

or $\log 0.9^m < \log 0.647\ 004$

$$m\log 0.9 < \log 0.647\ 004$$
$$m > 4.1325...$$

Since *m* is an integer, the minimum value of *m* is 5.

∴ The factory will be reformed in the 5th year since 2011, i.e. in 2015.

40. (a) (i)

$$\text{Perimeter of } \triangle A_1B_1C_1 = (7+5+4)\text{ cm} = 16\text{ cm}$$

$$\text{Perimeter of } \triangle A_2B_2C_2 = \frac{1}{2}(7+5+4)\text{ cm} = 8\text{ cm}$$

$$\text{Perimeter of } \triangle A_3B_3C_3 = \frac{1}{2}\times\frac{1}{2}(7+5+4)\text{ cm}$$
$$=4\text{ cm}$$

∴ The perimeters of $\triangle A_1B_1C_1$, $\triangle A_2B_2C_2$, $\triangle A_3B_3C_3$, … form a geometric sequence with common ratio $\frac{1}{2}$.

∴ The perimeter of $\triangle A_kB_kC_k$

$$=16\left(\frac{1}{2}\right)^{k-1}\text{ cm}$$
$$=2^4(2)^{1-k}\text{ cm}$$
$$=\underline{\underline{2^{5-k}\text{ cm}}}$$

(ii) The sum of the perimeters of all the triangles formed

$$=\frac{16}{1-\frac{1}{2}}\text{ cm}$$
$$=\underline{\underline{32\text{ cm}}}$$

(b) (i) $s = \frac{7+5+4}{2}$ cm $= 8$ cm

By Heron's formula,

$$\text{area of } \triangle A_1B_1C_1 = \sqrt{8(8-7)(8-5)(8-4)} \text{ cm}^2$$
$$= 4\sqrt{6} \text{ cm}^2$$

$\because$ $\triangle A_1B_1C_1$, $\triangle A_2B_2C_2$, $\triangle A_3B_3C_3$, … are similar triangles.

$\therefore$ The areas of $\triangle A_1B_1C_1$, $\triangle A_2B_2C_2$, $\triangle A_3B_3C_3$, … form a geometric sequence with common ratio $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$.

$\therefore$

$$\text{Area of } \triangle A_2B_2C_2 = 4\sqrt{6}\left(\frac{1}{4}\right) \text{ cm}^2$$
$$= \underline{\underline{\sqrt{6} \text{ cm}^2}}$$

$$\text{Area of } \triangle A_3B_3C_3 = \sqrt{6}\left(\frac{1}{4}\right) \text{ cm}^2$$
$$= \underline{\underline{\frac{\sqrt{6}}{4} \text{ cm}^2}}$$

(ii) The area of $\triangle A_kB_kC_k$

$$= 4\sqrt{6}\left(\frac{1}{4}\right)^{k-1} \text{ cm}^2$$
$$= 4\sqrt{6}(4)^{1-k} \text{ cm}^2$$
$$= \underline{\underline{\sqrt{6}(4^{2-k}) \text{ cm}^2}}$$

(iii) The sum of the areas of all the triangles formed

$$= \frac{4\sqrt{6}}{1-\frac{1}{4}} \text{ cm}^2$$
$$= \frac{16\sqrt{6}}{3} \text{ cm}^2$$
$$\approx 13.0639 \text{ cm}^2$$
$$< 15 \text{ cm}^2$$

$\therefore$ Tony's claim is correct.

41. (a)

$$OA_2 = OB_1 \cos\theta$$
$$= OA_1 \cos\theta$$
$$= \underline{\underline{k\cos\theta}}$$
$$OA_3 = OB_2 \cos\theta$$
$$= OA_2 \cos\theta$$
$$= \underline{\underline{k\cos^2\theta}}$$

(b) (i)

$$\overset{\frown}{A_1B_1} = 2\pi(OA_1)\frac{\theta}{360^\circ}$$
$$= \underline{\underline{\frac{k\pi\theta}{180^\circ}}}$$
$$\overset{\frown}{A_2B_2} = 2\pi(OA_2)\frac{\theta}{360^\circ}$$
$$= \underline{\underline{\frac{k\pi\theta\cos\theta}{180^\circ}}}$$

$$\overset{\frown}{A_3B_3} = 2\pi(OA_3)\frac{\theta}{360^\circ}$$
$$= \underline{\underline{\frac{k\pi\theta\cos^2\theta}{180^\circ}}}$$

$$\overset{\frown}{A_nB_n} = \frac{k\pi\theta\cos^{n-1}\theta}{180^\circ}$$

(ii)
$$\frac{\overset{\frown}{A_nB_n}}{\overset{\frown}{A_{n-1}B_{n-1}}} = \frac{\dfrac{k\pi\theta\cos^{n-1}\theta}{180^\circ}}{\dfrac{k\pi\theta\cos^{n-2}\theta}{180^\circ}}$$
$$= \cos\theta, \text{ which is a constant.}$$

∴ $\overset{\frown}{A_1B_1}, \overset{\frown}{A_2B_2}, \overset{\frown}{A_3B_3}, \dots$ form a geometric sequence with common ratio $\cos\theta$.

∴
$$\overset{\frown}{A_1B_1} + \overset{\frown}{A_2B_2} + \overset{\frown}{A_3B_3} + \dots$$
$$= \frac{\dfrac{k\pi\theta}{180^\circ}}{1-\cos\theta}$$
$$= \frac{k\pi\theta}{180^\circ(1-\cos\theta)}$$

(c) Area of $\triangle OA_2B_1$
$$= \frac{1}{2}(OB_1)(OA_2)\sin\theta$$
$$= \frac{1}{2}(k)(k\cos\theta)\sin\theta$$
$$= \underline{\underline{\frac{1}{2}k^2\cos\theta\sin\theta}}$$

Area of $\triangle OA_3B_2$
$$= \frac{1}{2}(OB_2)(OA_3)\sin\theta$$
$$= \frac{1}{2}(k\cos\theta)(k\cos^2\theta)\sin\theta$$
$$= \underline{\underline{\frac{1}{2}k^2\cos^3\theta\sin\theta}}$$

Area of $\triangle OA_4B_3$
$$= \frac{1}{2}(OB_3)(OA_4)\sin\theta$$
$$= \frac{1}{2}(k\cos^2\theta)(OB_3\cos\theta)\sin\theta$$
$$= \frac{1}{2}(k\cos^2\theta)(k\cos^2\theta\cos\theta)\sin\theta$$
$$= \underline{\underline{\frac{1}{2}k^2\cos^5\theta\sin\theta}}$$

(d) The areas of the triangles form a geometric sequence with first term $\frac{1}{2}k^2\cos\theta\sin\theta$ and common ratio $\cos^2\theta$.

∴ (Area of $\triangle OA_2B_1$) + (area of $\triangle OA_3B_2$) + (area of $\triangle OA_4B_3$) + ...
$$= \frac{\frac{1}{2}k^2\cos\theta\sin\theta}{1-\cos^2\theta}$$
$$= \frac{k^2\cos\theta\sin\theta}{2\sin^2\theta}$$
$$= \frac{k^2\cos\theta}{2\sin\theta}$$
$$= \frac{k^2}{2\tan\theta}$$

Multiple Choice Questions (p. 2.49)

1. Answer: C

Number of dots in the 1st pattern = 2
Number of dots in the 2nd pattern = 2 + 3 = 5
Number of dots in the 3rd pattern = 5 + 5 = 10
Number of dots in the 4th pattern = 10 + 7 = 17
Number of dots in the 5th pattern = 17 + 9 = 26
Number of dots in the 6th pattern = 26 + 11 = 37

∴ Total number of dots in the first 6 patterns
$$= 2 + 5 + 10 + 17 + 26 + 37$$
$$= \underline{\underline{97}}$$

2. Answer: C

Let a and d be the first term and the common difference of the sequence respectively.

∵ $S(5) = 40$

∴
$$\frac{5}{2}[2a + (5-1)d] = 40$$
$$2a + 4d = 16$$
$$a + 2d = 8 \quad \text{......(1)}$$
$$T(9) = a + 8d = -16 \quad \text{......(2)}$$
$$(2) - (1):\ 6d = -24$$
$$d = -4$$

By substituting $d = -4$ into (1), we have
$$a + 2(-4) = 8$$
$$a = 16$$

∴ The first term of the sequence is 16.

3. Answer: B

First term $= 3(1) + 2 = 5$

Common difference $=[3(n+1)+2]-(3n+2)=3$

$$S(k)>5000$$
$$\frac{k}{2}[2(5)+(k-1)(3)]>5000$$
$$7k+3k^2>10\ 000$$
$$3k^2+7k-10\ 000>0$$
$$k>56.5801...$$
$$\text{or } k<-58.9134...\text{(rejected)}$$

∴ The smallest value of k is 57.

4. Answer: D

$1, 9^2, 9^4, \ldots, 9^{2n}$ form a geometric sequence.

Common ratio $=\frac{9^2}{1}=9^2$

Number of terms $=\frac{2n}{2}+1=n+1$

$$\therefore \quad 1+9^2+9^4+\ldots+9^{2n}=\frac{1[(9^2)^{n+1}-1]}{9^2-1}$$
$$=\frac{81^{n+1}-1}{80}$$

5. Answer: D

Let a and r be the first term and the common ratio of the sequence respectively.

$$T(2)\times T(3)=-72$$
$$ar(ar^2)=-72$$
$$a^2r^3=-72 \quad ......(1)$$
$$T(3)\times T(5)=576$$
$$ar^2(ar^4)=576$$
$$a^2r^6=576 \quad ......(2)$$
$$\frac{(2)}{(1)}: \quad \frac{a^2r^6}{a^2r^3}=\frac{576}{-72}$$
$$r^3=-8$$
$$r=-2$$

By substituting $r=-2$ into (1), we have

$$a^2(-2)^3=-72$$
$$a^2=9$$
$$a=3 \text{ or } a=-3\text{ (rejected)}$$
$$\therefore \quad S(7)=\frac{3[1-(-2)^7]}{1-(-2)}$$
$$=129$$

6. Answer: C

First term $=a$ and common ratio $=\frac{-1}{a}=-\frac{1}{a}$

$$S(\infty)=\frac{a}{1-\left(-\frac{1}{a}\right)}$$
$$\therefore \quad =\frac{a}{1+\frac{1}{a}}$$
$$=\frac{a}{\frac{a+1}{a}}$$
$$=\frac{a^2}{a+1}$$

7. Answer: B

Consider the geometric sequence $-4, -1, -\frac{1}{4}, \ldots$

First term = –4 and common ratio $=\frac{-1}{-4}=\frac{1}{4}$

∴ The sum of all the negative terms

$$=\frac{-4}{1-\frac{1}{4}}$$
$$=-\frac{16}{3}$$

8. Answer: B

Total interest at the end of the 3rd year

$$=\$\left[\frac{500\left(1+\frac{4\%}{12}\right)\left[\left(1+\frac{4\%}{12}\right)^{3(12)}-1\right]}{\left(1+\frac{4\%}{12}\right)-1}-500(3)(12)\right]$$
$$=\$1154 \text{ (cor. to the nearest dollar)}$$

9. Answer: D

Let a and r be the first term and the common ratio of the sequence respectively.

$$\because \quad S(2)=\frac{25}{3}$$
$$\therefore \quad a+ar=\frac{25}{3} \quad ......(1)$$
$$\because \quad S(\infty)=15$$
$$\therefore \quad \frac{a}{1-r}=15 \quad ......(2)$$
$$\frac{(1)}{(2)}: 1-r^2=\frac{25}{3}\left(\frac{1}{15}\right)$$
$$1-r^2=\frac{5}{9}$$
$$r^2=\frac{4}{9}$$
$$r=-\frac{2}{3} \text{ or } r=\frac{2}{3}$$

∴ The common ratio of the sequence is $-\frac{2}{3}$ or $\frac{2}{3}$.

10. Answer: B

For I,

$$T(n) = S(n) - S(n-1)$$
$$= (3n^2 - 2n) - [3(n-1)^2 - 2(n-1)]$$
$$= (3n^2 - 2n) - [3(n^2 - 2n + 1) - 2n + 2]$$
$$= 6n - 5$$

∴ I is true.

For II,

From I,

$$T(30) - T(31) = 6(30) - 5 - [6(31) - 5] = -6$$

∴ II is false.

For III,

$$T(6) + T(7) + ... + T(13) = S(13) - S(5)$$
$$= [3(13)^2 - 2(13)] - [3(5)^2 - 2(5)]$$
$$= 416$$
$$> 400$$

∴ III is true.

∴ The answer is B.

11. Answer: B

$$10 \times 10^2 \times 10^3 \times \ldots \times 10^n > 10^{50}$$
$$10^{1+2+3+\ldots+n} > 10^{50}$$
$$1+2+3+\ldots+n > 50$$
$$\frac{n(n+1)}{2} > 50$$
$$n(n+1) > 100$$
$$n^2 + n - 100 > 0$$
$$n > \frac{-1+\sqrt{401}}{2} \quad \text{or} \quad n < \frac{-1-\sqrt{401}}{2} \text{ (rejected)}$$
$$n > \frac{-1+\sqrt{401}}{2} = 9.5124...$$

∴ The smallest integral value of n is 10.

12. Answer: D

First term = $\sin^2\theta$ and common ratio = $\cos^2\theta$

$$S(\infty) = \frac{\sin^2\theta}{1-\cos^2\theta}$$
$$= \frac{\sin^2\theta}{\sin^2\theta}$$
$$= \underline{\underline{1}}$$

13. Answer: C

$$a^2 = 27b \quad ......(1)$$
$$2b = 15 + a$$
$$b = \frac{a}{2} + \frac{15}{2} \quad ......(2)$$

By substituting (2) into (1), we have

$$a^2 = 27\left(\frac{a}{2} + \frac{15}{2}\right)$$
$$2a^2 - 27a - 405 = 0$$
$$(a+9)(2a-45) = 0$$
$$a = -9 \text{ or } a = \frac{45}{2} \text{ (rejected)}$$

When $a = -9$,

common ratio $= \frac{-9}{27} = -\frac{1}{3}$

∴ The sum to infinity

$$= \frac{27}{1-\left(-\frac{1}{3}\right)}$$

$$= \underline{\underline{\frac{81}{4}}}$$

HKMO (p. 2.50)

1.

$$P = a_2 + a_4 + \dots + a_{100}$$

$$P - 50 = a_2 + a_4 + \dots + a_{100} - 50$$

$$P - 50 = (a_2 - 1) + (a_4 - 1) + \dots + (a_{100} - 1)$$

$$= a_1 + a_3 + \dots + a_{99}$$

$$a_1 + a_2 + \dots + a_{100} = 2012$$

$$(a_1 + a_3 + \dots + a_{99}) + (a_2 + a_4 + \dots + a_{100}) = 2012$$

$$(P - 50) + P = 2012$$

$$2P = 2062$$

$$P = \underline{\underline{1031}}$$

2. $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \dots$ form a geometric sequence with first term = 1 and common ratio $= \frac{\frac{1}{3}}{1} = \frac{1}{3}$.

$$\log_4 N = 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \dots$$

$$\log_4 N = \frac{1}{1 - \frac{1}{3}}$$

$$\log_4 N = \frac{3}{2}$$

$$N = 4^{\frac{3}{2}}$$

$$= \underline{\underline{8}}$$

3. Let a and d be the number of seats in the first row and the common difference of the sequence respectively.

$$d = 2$$

$$a + \frac{31 - 1}{2}(2) = 64$$

$$a + 30 = 64$$

$$a = 34$$

∴ The total number of seats in the concert hall

$$= \frac{31}{2}[2(34) + (31 - 1)(2)]$$

$$= \underline{\underline{1984}}$$

4.

$$p = 2 - 2^2 - 2^3 - 2^4 - \dots - 2^{10} + 2^{11}$$

$$= 2 + 2^{11} - (2^2 + 2^3 + 2^4 + \dots + 2^{10})$$

$$= 2050 - \frac{2^2(2^9 - 1)}{2 - 1}$$

$$= \underline{\underline{6}}$$

5.

$$F = 1 + 2 + 2^2 + 2^3 + \dots + 2^{120}$$

$$= \frac{1(2^{121} - 1)}{2 - 1}$$

$$= 2^{121} - 1$$

$$T = \sqrt{\frac{\log(1+F)}{\log 2}}$$

$$= \sqrt{\frac{\log(1 + 2^{121} - 1)}{\log 2}}$$

$$= \sqrt{\frac{\log 2^{121}}{\log 2}}$$

$$= \sqrt{\frac{121 \log 2}{\log 2}}$$

$$= \sqrt{121}$$

$$= \underline{\underline{11}}$$

6. Consider the triangular numbers.
The kth triangular number is

$$1 + 2 + 3 + \dots + k = \frac{k(k+1)}{2}.$$

Let $T(k)$ be the largest triangular number that is smaller than or equal to 2003.

∵ $T(k) \le 2003$

∴

$$\frac{k(k+1)}{2} \le 2003$$

$$k^2 + k \le 4006$$

$$k^2 + k - 4006 \le 0$$

$$\frac{-1 - \sqrt{1^2 - 4(1)(-4006)}}{2(1)} \le k \le \frac{-1 + \sqrt{1^2 - 4(1)(-4006)}}{2(1)}$$

$$\frac{-1 - \sqrt{16\,025}}{2} \le k \le \frac{-1 + \sqrt{16\,025}}{2}$$

$$\frac{-1 - 5\sqrt{641}}{2} \le k \le \frac{-1 + 5\sqrt{641}}{2}$$

∴ The largest value of k is 62.

∴ The largest triangular number that is smaller than or equal to 2003

$$= \frac{62(62+1)}{2}$$

$$= 1953$$

∴ 2003 is in row (2003 – 1953) = 50 and column [62 + 1 – (50 – 1)] = 14.

∴ $x = 50$ and $y = 14$

∴ $xy = (50)(14) = \underline{\underline{700}}$

Exam Focus

Exam-type Questions (p. 2.52)

1. (a) ∵ The revenue in 2015 is \$1 936 000.

$$\$1\,600\,000(1+k\%)^{3-1} = \$1\,936\,000$$

$$\therefore \quad \left(1+\frac{k}{100}\right)^2 = 1.21$$

$$1+\frac{k}{100} = 1.1$$

$$k = \underline{\underline{10}}$$

(b) Let the total revenue exceed \$20 000 000 in the *n*th year since 2013.

$$\$1\,600\,000[1+(1+10\%)+\ldots+(1+10\%)^{n-1}] + \$900\,000[1+(1+21\%)+\ldots+(1+21\%)^{n-3}] > \$20\,000\,000$$

$$\frac{16[(1.1)^n-1]}{1.1-1}+\frac{9[(1.21)^{n-2}-1]}{1.21-1} > 200$$

$$160(1.1^n-1)+\frac{300}{7}(1.1^{2n-4}-1) > 200$$

$$\frac{300}{7(1.1)^4}(1.1)^{2n}+160(1.1)^n-\frac{2820}{7} > 0$$

$\therefore \quad 1.1^n > 1.874\,806\,311$ or

$1.1^n < -7.340\,779\,644$ (rejected)

Consider $1.1^n > 1.874\,806\,311$

$$n\log 1.1 > \log 1.874\,806\,311$$

$$n > 6.594\,315\,052$$

Since *n* is an integer, the minimum value of *n* is 7.

∴ The total revenue first exceeds \$20 000 000 in the 7th year since 2013, i.e. in 2019.

2. (a) (i) Number of cards in the 1st row = 1 = 2(1) – 1
Number of cards in the 2nd row = 3 = 2(2) – 1
Number of cards in the 3rd row = 5 = 2(3) – 1
∴ Number of cards in the 10th row

$$= 2(10)-1$$
$$= \underline{\underline{19}}$$

(ii) Total number of cards in the first 9 rows

$$= \frac{9}{2}[2(1)+(9-1)(2)] = 81$$

∴ The smallest number in the 10th row
= 81 + 1
= 82

∴ Jimmy's claim is not correct.

(b) (i) Total number of cards in the first 10 rows

$$= \frac{10}{2}[2(1)+(10-1)(2)]$$
$$= 100$$

The sum of numbers in the first 10 rows

$$= \frac{100}{2}(1+100)$$
$$= 5050$$

Total number of cards in the first 11 rows

$$= \frac{11}{2}[2(1)+(11-1)(2)]$$
$$= 121$$

The sum of numbers in the first 11 rows

$$= \frac{121}{2}(1+121)$$
$$= 7381$$

∴ The sum of numbers in the 11th row

$$= 7381-5050$$
$$= \underline{\underline{2331}}$$

(ii) Largest number in the 1st row = 1 = 1^2
Largest number in the 2nd row = 4 = 2^2
Largest number in the 3rd row = 9 = 3^2
∴ Largest number in the *k*th row = k^2
∴ The required sum

$=4^2+5^2+6^2+7^2+8^2+9^2+10^2+11^2$
$=\underline{\underline{492}}$

3. **(a)** **(i)** ∵ T_1, T_2, T_3, ... are similar solid.
∴ The volumes of T_1, T_2, T_3, … form a geometric sequence with first term = $(80^2)(40)\pi$ cm^3 = 256 000 π cm^3 and common ratio $=\left(\frac{3}{4}\right)^3=\frac{27}{64}$.
∴

$$\text{Volume of } T_8 = 256000\pi\left(\frac{27}{64}\right)^{8-1}\text{ cm}^2$$
$$=\underline{\underline{1913\text{ cm}^2}}\text{ (cor. to the nearest integer)}$$

(ii) ∵ Common ratio = $\frac{27}{64}$, which is a constant
∴ Kelvin's claims is correct.

(b) **(i)** Suppose k cylinders can be made.

$$1.3\times100^3\text{ cm}^2 > 256000\pi\left[1+\frac{27}{64}+\ldots+\left(\frac{27}{64}\right)^{k-1}\right]$$
$$\frac{1-\left(\frac{27}{64}\right)^k}{1-\frac{27}{64}} < \frac{325}{64\pi}$$
$$1-\left(\frac{27}{64}\right)^k < \frac{12025}{4096\pi}$$
$$\left(\frac{27}{64}\right)^k > \frac{4096\pi-12025}{4096\pi}$$
$$\log\left(\frac{27}{64}\right)^k > \log\frac{4096\pi-12025}{4096\pi}$$
$$k\log\frac{27}{64} > \log\frac{4096\pi-12025}{4096\pi}$$
$$k < \frac{\log\frac{4096\pi-12025}{4096\pi}}{\log\frac{27}{64}}$$
$$k < 3.1580\ldots$$

Since k is an integer, the maximum value of k is 3.
∴ At most 3 cylinders can be made.

(ii) Total surface area of T_1
$$=[2(80^2\pi)+2(80)(40)\pi]\text{ cm}^2$$
$$=19\,200\pi\text{ cm}^2$$
∵ T_1, T_2, T_3, ... are similar solid.
∴

$$\frac{\text{Surface area of } T_2}{\text{Surface area of } T_1} = \left(\frac{3}{4}\right)^2$$

$$\text{Surface area of } T_2 = \left(\frac{3}{4}\right)^2 \times \text{surface are}$$

$$= \left(\frac{3}{4}\right)^2 \times 19\ 200\pi \text{ cr}$$

$$= 10\ 800\pi \text{ cm}^2$$

$$\frac{\text{Surface area of } T_3}{\text{Surface area of } T_2} = \left(\frac{3}{4}\right)^2$$

$$\text{Surface area of } T_3 = \left(\frac{3}{4}\right)^2 \times \text{surface are}$$

$$= \left(\frac{3}{4}\right)^2 \times 10\ 800\pi \text{ cr}$$

$$= 6075\pi \text{ cm}^2$$

∴ Total cost of painting

$$= \$0.02(19\ 200\pi + 10\ 800\pi + 6075\pi)$$
$$= \$0.02(36\ 075\pi)$$
$$= \underline{\underline{\$2267}} \text{ (cor. to the nearest dollar)}$$

4. **(a)** **(i)** The loan interest for the 1st month

$$= \$100\ 000\left(\frac{12\%}{12}\right)$$
$$= \$100\ 000(0.01)$$
$$= \underline{\underline{\$1000}}$$

(ii) The amount owed after paying the 2nd instalment

$$= \$\{[100\ 000(1.01) - k](1.01) - k\}$$
$$= \$[100\ 000(1.01)^2 - 2.01k]$$
$$= \underline{\underline{\$(102\ 010 - 2.01k)}}$$

(b) The amount owed after paying the nth instalment

$$= \$[100000(1.01)^n - (1.01)^{n-1}k - (1.01)$$
$$= \$\left[100000(1.01)^n - \frac{k(1.01^{n-1} - 1)}{1.01 - 1}\right]$$
$$= \$\{100000(1.01)^n - 100k[(1.01^{n-1} - 1)]$$

5. **Answer: C**

Let d be the common difference of the sequence.

$a_4 = a_1 + 3d = 20$ ......(1)

$a_9 = a_1 + 8d = 55$ ......(2)

(2) – (1):
$$a_1 + 8d - (a_1 + 3d) = 55 - 20$$
$$5d = 35$$
$$d = 7$$

By substituting $d = 7$ into (1), we have

$$a_1 + 3(7) = 20$$
$$a_1 = -1$$

∴

$$a_{11} + a_{12} + \ldots + a_{20}$$
$$= S(20) - S(10)$$
$$= \frac{20}{2}[2(-1) + (20 - 1)(7)] - \frac{10}{2}[2(-1) + (10$$
$$= \underline{\underline{1005}}$$

6. Answer: B

$$\sin^2 1° + 2\sin^2 2° + ... + 44\sin^2 44° +$$
$$+ 44\sin^2 46° + ... + 2\sin^2 88° + \sin$$
$$= (\sin^2 1° + \sin^2 89°) + 2(\sin^2 2° + \sin$$
$$+ 44(\sin^2 44° + \sin^2 46°) + 45\sin^2$$
$$= (\sin^2 1° + \cos^2 1°) + 2(\sin^2 2° + \cos$$
$$+ 44(\sin^2 44° + \cos^2 44°) + 45\left(\frac{1}{\sqrt{2}}\right.$$
$$= 1 + 2 + ... + 44 + \frac{45}{2}$$
$$= \frac{44}{2}(1 + 44) + \frac{45}{2}$$
$$= \frac{2025}{2}$$

7. Answer: B

Let r be the common ratio of the sequence.

$$3r^7 = 384$$
$$r^7 = 128$$
$$r = 2$$

$$\therefore \quad a + b + c + d + e + f = \frac{3(2^7 - 1)}{2 - 1} - 3$$
$$= 378$$

8. Answer: D

Let a and r be the first term and the common ratio of the sequence respectively.

$$\because \quad r = \frac{y}{x} \text{ and } a = \frac{x}{\frac{y}{x}} = \frac{x^2}{y}$$

$\therefore$ The sum to infinity

$$= \frac{\frac{x^2}{y}}{1 - \frac{y}{x}}$$
$$= \frac{x^2}{y} \times \frac{x}{x - y}$$
$$= \frac{x^3}{y(x - y)}$$